49. Let $U$ and $V$ be vector spaces and $L : U \to V$ be a linear transformation. Show that

    (a) $\mathcal{N}(L) = \{u \in U : \; L(u) = 0\}$ is a subspace of $U$;

    (b) $\mathcal{R}(L) = \{L(u) : \; u \in U\}$ is a subspace of $V$.

50. Let $P_n$ be the set of all polynomials of degree smaller or equal to $n \in \mathbb{N}_0$. We know that $P_n$ is a vector space. For $p(t) = \sum_{k=0}^{n} a_k t^k$, define $v(p) = \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$. Let $q(t) = 1 - 4t^2 + 3t^3$.

    (a) For $q \in P_3$, find $v(q)$.

    (b) Find a basis and the dimension of $P_3$. Find $v(b)$ for each element $b$ in the basis.

    (c) Is $L(p) = p^2$ with $L : P_3 \to P_6$ a linear transformation?

    (d) Is $L(p) = p + q$ with $L : P_3 \to P_3$ a linear transformation?

51. For the following transformations $L : P_3 \to P_n$ do the following: Pick $n$. Find $L(q)$, where $q$ is given in the previous problem. Find $v(L(q))$. Show that $L$ is a linear transformation. Find a matrix $A$ such that $v(L(p)) = Av(p)$ for all $p \in P_3$. Find $\mathcal{N}(L)$ and $\mathcal{R}(L)$.

    (a) $L(p) = p'$;

    (b) $L(p) = pq$;

    (c) $L(p) = p''$;

    (d) $L(p)$ is the solution of the problem $x' = p$, $x(0) = 0$.

52. Find the lengths, the inner product, and the angle between

    (a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ (also draw a picture); (b) $\begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \\ 1 \\ 3 \end{bmatrix}$.

53. Find all vectors orthogonal to $\mathcal{R}(A)$, and all vectors orthogonal to $\mathcal{N}(A)$, if

    (a) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$; (b) $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$.

54. Find all vectors orthogonal to

    (a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; (b) $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

55. Prove the following statements where $x, y, z \in R^n$. Draw a picture for $n = 2$.

    (a) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.

    (b) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$.

    (c) $(x - y) \perp (x + y)$ iff $\|x\| = \|y\|$.

    (d) $(x - z) \perp (y - z)$ iff $\|x - z\|^2 + \|y - z\|^2 = \|x - y\|^2$.

    (e) $\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.