Instructions: There are 11 problems, each is worth 20 points. For each problem, there is dedicated space available. You **must** fit your complete solution in this space. You may do some preliminary calculations on an extra sheet of paper, but only your solution written on the dedicated space corresponding to the particular problem will be graded for credit. If there are boxes, put only the solution in the box. You have to do your work for boxes on separate sheets of paper, which will not be collected and not graded. For the boxes, either it is true or false, no partial credit will be given. If there are no boxes, you have to explain your solution in detail, but it still has to fit in the particular space. You may only use your calculator and something to write. Paper will be provided, you may not use your own paper. You have to leave the exam stapled together at all times. Violations of these instructions will result in point deductions. Any form of cheating or any talking to some other student during the exam will result in immediate collection of that exam, **without** warning (well, this is the warning).

Problem 1: Please complete the following truth tables:

p	q	$p \vee q$	$\overline{p} \vee q$	$(p \vee q) \wedge \overline{\overline{p} \vee q}$
T	F			
F	F			
F	T			
T	T			

p	q	$p \wedge q$	$\overline{q}$	$(p \wedge q) \vee \overline{q}$
T	F			
F	F			
F	T			
T	T			

Problem 2: Write "T" for "true" or "F" for "false" in the boxes:

1. $\forall x \in \mathbb{R}\ x^2 - 9 = 0$ is [ ]

2. $\exists x \in \mathbb{R}\ x^2 - 9 = 0$ is [ ]

3. $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ xy = 0$ is [ ]

4. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ x = y^2$ is [ ]

Problem 3: Please prove the following statements using the Principle of Mathematical Induction:

1. $\forall n \in \mathbb{N} \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

2. $\forall n \in \mathbb{N} \setminus \{1, 2\}\ 2n + 1 \leq 2^n$

Problem 4: Let $P(n)$: $\sum_{k=1}^{n}(2k) = (n+2)(n-1)$. Please find the truth values of the following propositions (and prove your claims):

1. $\forall n \in \mathbb{N}\ P(n) \rightarrow P(n+1)$

2. $\overline{\forall n \in \mathbb{N}\ P(n)}$

Problem 5: Let $a_n = \frac{1}{n} - \frac{1}{n+1}$, $n \in \mathbb{N}$.

1. Find $\sum_{k=1}^{100} a_k$ and $\prod_{k=1}^{100} a_k$

2. Is a increasing or decreasing? Prove your claim.

<u>Problem 6:</u> Let $f_1 = 1$, $f_2 = 2$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

1. Please enter the values in the boxes: $\boxed{f_3 = \quad\quad}$, $\boxed{f_4 = \quad\quad}$, $\boxed{f_6 = \quad\quad}$.
2. Show that $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1} - 1$ holds for all $n \in \mathbb{N}$
3. Show that $\sum_{k=1}^{n} f_k = f_{n+2} - 2$ holds for all $n \in \mathbb{N}$

<u>Problem 7:</u> Let $a_0 = 2$ and $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$ for $n \in \mathbb{N}$. Please show that $\{a_n\}$ converges and compute its limit.

Problem 8: Let $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $Z = \{z_1, z_2, z_3\}$, and define $f : X \to Y$, $g : Y \to Z$, $h : Z \to X$ by $f(x_1) = y_1$, $f(x_2) = y_3$, $f(x_3) = y_4$, $f(x_4) = y_2$, $g(y_1) = z_1$, $g(y_2) = z_1$, $g(y_3) = z_3$, $g(y_4) = z_2$, $h(z_1) = x_1$, $h(z_2) = x_2$, and $h(z_3) = x_4$.

1. Write "y" for "yes" and "n" for "no" in the boxes: f is one-to-one [:]; f is onto [:]; h is one-to-one [:]; h is onto [:]; g is one-to-one [:]; g is onto [:].
2. Please find $h \circ g \circ f$.

Problem 9: This exercise refers to a club consisting of six distinct men and seven distinct women. In how many ways can we select a committee consisting of (enter the number values)

1. three men and four women? ⁚ [ ];
2. five persons? ⁚ [ ];
3. four persons that has at most one man? ⁚ [ ];
4. four persons that has persons of both sexes? ⁚ [ ];
5. four persons that has at least one woman? ⁚ [ ].
6. Please explain your solution for number 4.

Problem 10: Please find the solutions to the following initial value problems. Show all your work.

1. $a_n = 2na_{n-1}\ \forall n \in \mathbb{N},\ a_0 = 1$

2. $b_{n+1} = 7b_n - 10b_{n-1}\ \forall n \in \mathbb{N},\ b_0 = 5,\ b_1 = 16$

3. $L_{n+2} = L_{n+1} + L_n\ \forall n \in \mathbb{N},\ L_1 = 1,\ L_2 = 3$

Problem 11: Suppose it costs \$10,000 to purchase a new car. The annual operating cost for a car during it's first year is \$300, during the second year \$500, during the third year \$800, during the fourth year \$1200, during the fifth year \$1600, and during the sixth year \$2200. The resale value of a one year old car is \$7000, of a two year old car \$6000, for a three year old car \$4000, for a four year old car \$3000, for a five year old car \$2000, and for a six year old car \$1000. Assuming that one has a new car at present, determine a replacement policy that minimizes the net costs of owning and operating a car for the next six years. Please draw a graph and use the Dijkstra Algorithm to work on this problem. Also enter the minimal net cost in here: