Introduction to Real Analysis (Math 315)

Spring 2005 Lecture Notes

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CHAPTER 1

The Riemann–Stieltjes Integral

1.1. Functions of Bounded Variation

Definition 1.1. Let $a, b \in \mathbb{R}$ with $a < b$. A *partition* P of $[a, b]$ is a finite set of points $\{x_0, x_1, \ldots, x_n\}$ with

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

The set of all partitions of $[a, b]$ is denoted by $\mathcal{P} = \mathcal{P}[a, b]$. If $P \in \mathcal{P}$, then the *norm* of $P = \{x_0, x_1, \ldots, x_n\}$ is defined by

$$\|P\| = \sup_{1 \le i \le n} \Delta x_i, \quad \text{where} \quad \Delta x_i = x_i - x_{i-1}, \quad 1 \le i \le n.$$

Definition 1.2. Let $f : [a, b] \to \mathbb{R}$ be a function. We put

$$\bigvee(P, f) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \quad \text{for} \quad P = \{x_0, \ldots, x_n\} \in \mathcal{P}.$$

The *total variation* of f on $[a, b]$ is defined as

$$\bigvee_a^b f = \sup_{P \in \mathcal{P}} \bigvee(P, f).$$

If $\bigvee_a^b f < \infty$, then f is said to be of *bounded variation* on $[a, b]$. We write $f \in \mathrm{BV}[a, b]$.

Example 1.3. If f is nondecreasing on $[a, b]$, then $f \in \mathrm{BV}[a, b]$.

Theorem 1.4. *If $f' \in \mathrm{B}[a, b]$, then $f \in \mathrm{BV}[a, b]$.*

Theorem 1.5. $\mathrm{BV}[a, b] \subset \mathrm{B}[a, b]$.

1.2. The Total Variation Function

Lemma 1.6. $\mathrm{BV}[a, b] \subset \mathrm{BV}[a, x]$ *for all* $x \in (a, b)$.

Definition 1.7. For $f \in \mathrm{BV}[a, b]$ we define the *total variation function* $v_f : [a, b] \to \mathbb{R}$ by

$$v_f(x) = \bigvee_a^x f \quad \text{for all} \quad x \in [a, b].$$

Lemma 1.8. *If $f \in \mathrm{BV}[a, b]$, then v_f is nondecreasing on $[a, b]$.*

Lemma 1.9. *If $f \in \mathrm{BV}[a, b]$, then $v_f - f$ is nondecreasing on $[a, b]$.*

Theorem 1.10. *$f \in \mathrm{BV}[a, b]$ iff $f = g - h$ with on $[a, b]$ nondecreasing functions g and h.*

1.3. Riemann–Stieltjes Sums and Integrals

Definition 1.11. Let $f, g : [a, b] \to \mathbb{R}$ be functions. Let $P = \{x_0, \ldots, x_n\} \in \mathcal{P}[a, b]$ and $\xi = (\xi_1, \ldots, \xi_n)$ such that

$$x_{k-1} \le \xi_k \le x_k \quad \text{for all} \quad 1 \le k \le n.$$

Then

$$S(P, \xi, f, g) = \sum_{k=1}^{n} f(\xi_k)\,[g(x_k) - g(x_{k-1})]$$

is called a *Riemann–Stieltjes sum* for f with respect to g. The function f is called *Riemann–Stieltjes integrable* with respect to g over $[a, b]$, we write $f \in \mathcal{R}(g)$, if there exists a number J with the following property:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall P \in \mathcal{P},\ \|P\| < \delta :\ |S(P, \xi, f, g) - J| < \varepsilon$$

(independent of ξ). In this case we write

$$\int_a^b f dg = J,$$

and J is called the *Riemann–Stieltjes integral* of f with respect to g over $[a, b]$. The function f is also called *integrand* (function) while g is called *integrator* (function).

Theorem 1.12 (Fundamental Inequality). *If $f \in \mathrm{B}[a, b]$, $g \in \mathrm{BV}[a, b]$, and $f \in \mathcal{R}(g)$, then*

$$\left| \int_a^b f dg \right| \le \|f\|_\infty \bigvee_a^b g,$$

where $\|f\|_\infty = \sup_{a \le x \le b} |f(x)|$.

Example 1.13. If $g(x) = x$ for all $x \in [a, b]$, then $f \in \mathcal{R}(g)$ iff $f \in \mathrm{R}[a, b]$.

Example 1.14. $f \in \mathrm{C}[a, b]$ and $g(x) = \begin{cases} 0 & \text{if } a \le x \le t \\ p & \text{if } t < x \le b. \end{cases}$

Theorem 1.15. *Let $f \in \mathrm{R}[a, b]$ and $g' \in \mathrm{C}[a, b]$. Then*

$$f \in \mathcal{R}(g) \quad \text{and} \quad \int_a^b f dg = \int_a^b f g'.$$

Example 1.16. $\int_0^1 x d(x^2) = 2/3$.

Theorem 1.17. *If $f \in \mathcal{R}(g) \cap \mathcal{R}(h)$, then*

$$f \in \mathcal{R}(g + h) \quad \text{and} \quad \int f d(g + h) = \int f dg + \int f dh.$$

If $f \in \mathcal{R}(h)$ and $g \in \mathcal{R}(h)$, then

$$f + g \in \mathcal{R}(h) \quad \text{and} \quad \int (f + g) dh = \int f dh + \int g dh.$$

If $f \in \mathcal{R}(g)$ and $\rho \in \mathbb{R}$, then

$$\rho f \in \mathcal{R}(g),\ f \in \mathcal{R}(\rho g), \quad \text{and} \quad \int (\rho f) dg = \int f d(\rho g) = \rho \int f dg.$$

Lemma 1.18. *If $f \in \mathcal{R}(g)$ on $[a, b]$ and if $c \in (a, b)$, then $f \in \mathcal{R}(g)$ on $[a, c]$.*

Remark 1.19. Similarly, the assumptions of Lemma 1.18 also imply $f \in \mathcal{R}(g)$ on $[c,b]$. Also, we make the definition

$$\int_b^a f dg = -\int_a^b f dg \quad \text{if} \quad a < b.$$

Theorem 1.20. *If $f \in \mathcal{R}(g)$ on $[a,b]$ and if $c \in (a,b)$, then*

$$\int_a^c f dg + \int_c^b f dg = \int_a^b f dg.$$

Theorem 1.21. *If $f \in \mathcal{R}(g)$, then $g \in \mathcal{R}(f)$, and*

$$\int_a^b f dg + \int_a^b g df = f(b)g(b) - f(a)g(a).$$

Example 1.22. $\int_{-1}^2 x d|x| = 5/2$.

Theorem 1.23 (Main Existence Theorem)**.** *If $f \in \mathrm{C}[a,b]$ and $g \in \mathrm{BV}[a,b]$, then $f \in \mathcal{R}(g)$.*

1.4. Nondecreasing Integrators

Throughout this section we let $f \in \mathrm{B}[a,b]$ and α be a nondecreasing function on $[a,b]$.

Definition 1.24. If $P \in \mathcal{P}$, then we define the *lower and upper sums* L and U by

$$L(P,f,\alpha) = \sum_{k=1}^n m_k \left[\alpha(x_k) - \alpha(x_{k-1})\right], \quad m_k = \min_{x_{k-1} \le x \le x_k} f(x)$$

and

$$U(P,f,\alpha) = \sum_{k=1}^n M_k \left[\alpha(x_k) - \alpha(x_{k-1})\right], \quad M_k = \max_{x_{k-1} \le x \le x_k} f(x).$$

We also define the *lower and upper Riemann–Stieltjes integrals* by

$$\underline{\int_a^b} f d\alpha = \sup_{P \in \mathcal{P}} L(P,f,\alpha) \quad \text{and} \quad \overline{\int_a^b} f d\alpha = \inf_{P \in \mathcal{P}} U(P,f,\alpha).$$

Lemma 1.25. $\underline{\int_a^b} f d\alpha, \overline{\int_a^b} f d\alpha \in \mathbb{R}$.

Theorem 1.26. *If $P, P^* \in \mathcal{P}$ with $P^* \supset P$, then*

$$L(P,f,\alpha) \le L(P^*,f,\alpha) \quad \text{and} \quad U(P,f,\alpha) \ge U(P^*,f,\alpha).$$

Theorem 1.27. $\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha$.

Theorem 1.28. *If $f \in \mathrm{C}[a,b]$ and α is nondecreasing on $[a,b]$, then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.*

CHAPTER 2

Sequences and Series of Functions

2.1. Uniform Convergence

Definition 2.1. Let $\{f_n\}_{n\in\mathbb{N}}$ be funtions defined on $E \subset \mathbb{R}$. Suppose $\{f_n\}$ converges for all $x \in E$. Then f defined by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for} \quad x \in E$$

is called the *limit function* of $\{f_n\}$. We also say that $f_n \to f$ pointwise on E. If $f_n = \sum_{k=1}^n g_k$ for functions g_k, $k \in \mathbb{N}$, then f is also called the *sum* of the series $\sum_{k=1}^n g_k$, write $\sum_{k=1}^\infty g_k$.

Example 2.2. (i) $f_n(x) = 4x + x^2/n$, $x \in \mathbb{R}$.
(ii) $f_n(x) = x^n$, $x \in [0,1]$.
(iii) $f_n(x) = \lim_{m\to\infty} [\cos(n!\pi x)]^{2m}$, $x \in \mathbb{R}$.
(iv) $f_n(x) = \frac{\sin(nx)}{n}$, $x \in \mathbb{R}$.
(v) $f_n(x) = \begin{cases} 2n^2 x \text{ if } 0 \le x \le \frac{1}{2n} \\ 2n(1-nx) \text{ if } \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 \text{ if } \frac{1}{n} \le x \le 1. \end{cases}$

Definition 2.3. We say that $\{f_n\}_{n\in\mathbb{N}}$ converges *uniformly* on E to a function f if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N\ \forall x \in E: \ |f_n(x) - f(x)| \le \varepsilon.$$

If $f_n = \sum_{k=1}^n g_k$, we also say that the series $\sum_{k=1}^\infty g_k$ converges uniformly provided $\{f_n\}$ converges uniformly.

Example 2.4. (i) $f_n(x) = x^n$, $x \in [0, 1/2]$.
(ii) $f_n(x) = x^n$, $x \in [0,1]$.

Theorem 2.5 (Cauchy Criterion)**.** *The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on E iff*

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m,n \ge N\ \forall x \in E: \ |f_n(x) - f_m(x)| \le \varepsilon.$$

Theorem 2.6 (Weierstraß M-Test)**.** *Suppose $\{g_k\}_{k\in\mathbb{N}}$ satisfies*

$$|g_k(x)| \le M_k\ \forall x \in E\ \forall k \in \mathbb{N} \quad \textit{and} \quad \sum_{k=1}^\infty M_k \textit{ converges.}$$

Then $\sum_{k=1}^\infty g_k$ converges uniformly.

2.2. Properties of the Limit Function

Theorem 2.7. *Suppose $f_n \to f$ uniformly on E. Let x be a limit point of E and suppose*

$$A_n = \lim_{t\to x} f_n(t) \quad \textit{exists for all} \quad n \in \mathbb{N}.$$

Then $\{A_n\}_{n\in\mathbb{N}}$ *converges and*

$$\lim_{n\to\infty} A_n = \lim_{t\to x} f(t).$$

Theorem 2.8. *If f_n are continuous on E for all $n \in \mathbb{N}$ and $f_n \to f$ uniformly on E, then f is continuous on E.*

Corollary 2.9. *If g_k are continuous on E for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} g_k$ converges unifomly on E, then $\sum_{k=1}^{\infty} g_k$ is continuous on E.*

Definition 2.10. Let X be a metric space. By $\mathrm{C}(X)$ we denote the space of all complex-valued, continuous, and bounded functions on X. The *supnorm* of $f \in \mathrm{C}(X)$ is defined by

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \quad \text{for} \quad f \in \mathrm{C}(X).$$

Theorem 2.11. $(\mathrm{C}(X),\ d(f,g) = \|f-g\|_\infty)$ *is a complete metric space.*

Theorem 2.12. *Let α be nondecreasing on $[a,b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ for all $n \in \mathbb{N}$ and $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and*

$$\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n(x) d\alpha.$$

Corollary 2.13. *Let α be nondecreasing on $[a,b]$. Suppose $g_k \in \mathcal{R}(\alpha)$ on $[a,b]$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} g_k$ converges uniformly on $[a,b]$. Then*

$$\int_a^b \sum_{k=1}^{\infty} g_k d\alpha = \sum_{k=1}^{\infty} \int_a^b g_k d\alpha.$$

Theorem 2.14. *Let f_n be differentiable functions on $[a,b]$ for all $n \in \mathbb{N}$ such that $\{f_n(x_0)\}_{n\in\mathbb{N}}$ converges for some $x_0 \in [a,b]$. If $\{f_n'\}_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$, say to f, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad \textit{for all} \quad x \in [a,b].$$

Corollary 2.15. *Suppose g_k are differentiable on $[a,b]$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} g_k'$ is uniformly convergent on $[a,b]$. If $\sum_{k=1}^{\infty} g_k(x_0)$ converges for some point $x_0 \in [a,b]$, then $\sum_{k=1}^{\infty} g_k$ is uniformly convergent on $[a,b]$, and*

$$\left(\sum_{k=1}^{\infty} g_k\right)' = \sum_{k=1}^{\infty} g_k'.$$

2.3. Equicontinuous Families of Functions

Example 2.16. $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$, $0 \le x \le 1$, $n \in \mathbb{N}$.

Definition 2.17. A family $\mathcal{F}$ of functions defined on E is said to be *equicontinuous* on E if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \ (\forall x, y \in E:\ |x-y| < \delta)\ \forall f \in \mathcal{F}:\ |f(x) - f(y)| < \varepsilon.$$

Theorem 2.18. *Suppose K is compact, $f_n \in \mathrm{C}(K)$ for all $n \in \mathbb{N}$, and $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on K. Then the family $\mathcal{F} = \{f_n :\ n \in \mathbb{N}\}$ is equicontinuous.*

Definition 2.19. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is called *pointwise bounded* if there exists a function ϕ such that $|f_n(x)| < \phi(x)$ for all $n \in \mathbb{N}$. It is called *uniformly bounded* if there exists a number M such that $\|f_n\|_\infty \le M$ for all $n \in \mathbb{N}$.

Theorem 2.20. *A pointwise bounded sequence* $\{f_n\}_{n\in\mathbb{N}}$ *on a countable set* E *has a subsequence* $\{f_{n_k}\}_{k\in\mathbb{N}}$ *such that* $\{f_{n_k}(x)\}_{k\in\mathbb{N}}$ *converges for all* $x \in E$.

Theorem 2.21 (Arzelà–Ascoli)**.** *Suppose* K *is compact and* $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{C}(K)$ *is pointwise bounded and equicontinuous. Then* $\{f_n\}$ *is uniformly bounded on* K *and contains a subsequence which is uniformly convergent on* K.

2.4. Weierstraß' Approximation Theorem

Theorem 2.22 (Weierstraß' Approximation Theorem)**.** *Let* $f \in \mathrm{C}[a,b]$. *Then there exists a sequence of polynomials* $\{P_n\}_{n\in\mathbb{N}}$ *with*

$$P_n \to f \qquad \text{uniformly on} \qquad [a,b].$$

CHAPTER 3

Some Special Functions

3.1. Power Series

Definition 3.1. A function f is said to be represented by a *power series* around a provided

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for some} \quad c_n,\ n \in \mathbb{N}_0.$$

Such an f is called *analytic*.

Theorem 3.2. *If*

$$\sum_{n=0}^{\infty} c_n x^n \quad \textit{converges for} \quad |x| < R,$$

then it converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ *for all* $\varepsilon > 0$*. Also,* f *defined by*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \textit{for} \quad |x| < R$$

is continuous and differentiable on $(-R, R)$ *with*

$$f'(x) = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n \quad \textit{for} \quad |x| < R.$$

Corollary 3.3. *Under the hypotheses of Theorem 3.2,* $f^{(n)}$ *exists for all* $n \in \mathbb{N}$ *and*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1) c_n x^{n-k}$$

holds on $(-R, R)$*. In particular,*

$$f^{(n)}(0) = n! c_n \quad \textit{for all} \quad n \in \mathbb{N}_0.$$

Theorem 3.4 (Abel's Theorem)**.** *Suppose* $\sum_{n=0}^{\infty} c_n$ *converges. Put*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \textit{for} \quad x \in (-1, 1).$$

Then

$$\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} c_n.$$

Theorem 3.5. *Suppose* $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, *and* $\sum_{n=0}^{\infty} c_n$ *converge to* A, B, *and* C, *where* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$*. Then* $C = AB$*.*

Theorem 3.6. *Given a double sequence* $\{a_{ij}\}_{i,j\in\mathbb{N}}$. *If*

$$\sum_{j=1}^{\infty}|a_{ij}| = b_i \text{ for all } i\in\mathbb{N} \quad \text{and} \quad \sum_{i=1}^{\infty} b_i \text{ converges,}$$

then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Theorem 3.7 (Taylor's Theorem)**.** *Suppose* $\sum_{n=0}^{\infty} c_n x^n$ *converges in* $(-R, R)$ *and put* $f(x) = \sum_{n=0}^{\infty} c_n x^n$. *Then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{for} \quad |x-a| < R - |a|.$$

3.2. Exponential, Logarithmic, and Trigonometric Functions

Definition 3.8. We define the *exponential function* by

$$E(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \text{for all} \quad z\in\mathbb{C}.$$

Remark 3.9. In this remark, some properties of the exponential function are discussed.

Definition 3.10. We define the *trigonometric functions* by

$$C(x) = \frac{E(ix)+E(-ix)}{2} \quad \text{and} \quad S(x) = \frac{E(ix)-E(-ix)}{2i}.$$

Remark 3.11. In this remark, some properties of trigonometric functions are discussed.

3.3. Fourier Series

Definition 3.12. A *trigonometric polynomial* is a sum

$$f(x) = a_0 + \sum_{n=1}^{N}\left(a_n\cos(nx) + b_n\sin(nx)\right), \quad \text{where} \quad a_k, b_k \in \mathbb{C}.$$

A *trigonometric series* is a series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right).$$

Remark 3.13. In this remark, some properties of trigonometric series are discussed.

Definition 3.14. A sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is called an *orthogonal* system of functions on $[a, b]$ if

$$\langle\phi_n, \phi_m\rangle := \int_a^b \phi_n(x)\overline{\phi_m(x)}dx = 0 \quad \text{for all} \quad m \neq n.$$

If, in addition

$$\|\phi_n\|_2^2 := \langle\phi_n, \phi_n\rangle = 1 \quad \text{for all} \quad n\in\mathbb{N},$$

then $\{\phi_n\}$ is called *orthonormal* on $[a, b]$. If $\{\phi_n\}$ is *orthonormal* on $[a, b]$, then

$$c_n = \langle f, \phi_n\rangle \quad \text{for all} \quad n\in\mathbb{N}$$

are called the *Fourier coefficients* of a function f relative to $\{\phi_n\}$, and

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$$

is called the *Fourier series* of f.

Theorem 3.15. *Let $\{\phi_n\}$ be orthonormal on $[a, b]$. Let*

$$s_n(f;x) := s_n(x) := \sum_{m=1}^{n} c_m \phi_m(x), \quad \textit{where} \quad f(x) \sim \sum_{m=1}^{\infty} c_m \phi_m(x),$$

and put

$$t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x) \quad \textit{with} \quad \gamma_m \in \mathbb{C}.$$

Then

$$\|f - s_n\|_2^2 \leq \|f - t_n\|_2^2$$

with equality if $\gamma_m = c_m$ for all $m \in \mathbb{N}$.

Theorem 3.16 (Bessel's Inequality)**.** *If $\{\phi_n\}$ is orthonormal on $[a, b]$ and if $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$, then*

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \|f\|_2^2.$$

In particular, $\lim_{n\to\infty} c_n = 0$.

Definition 3.17. The *Dirichlet kernel* is defined by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} \quad \text{for all} \quad x \in \mathbb{R}.$$

Remark 3.18. In this remark, some properties of the Dirichlet kernel are discussed.

Theorem 3.19 (Localization Theorem)**.** *If, for some $x \in \mathbb{R}$, there exist $\delta > 0$ and $M < \infty$ with*

$$|f(x+t) - f(x)| \leq M|t| \quad \textit{for all} \quad t \in (-\delta, \delta),$$

then

$$\lim_{N\to\infty} s_N(f;x) = f(x).$$

Corollary 3.20. *If $f(x) = 0$ for all x in some interval J, then $s_N(f;x) \to 0$ as $N \to \infty$ for all $x \in J$.*

Theorem 3.21 (Parseval's Formula)**.** *If f and g have period 2π and are Riemann integrable,*

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \textit{and} \quad g(x) \sim \sum_{n=-\infty}^{\infty} \gamma_n e^{inx},$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx = \sum_{n=-\infty}^{\infty} c_n \overline{\gamma_n}, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

and

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x) - s_N(f;x)|^2 \, dx = 0.$$

3.4. The Gamma Function

Definition 3.22. For $x \in (0, \infty)$, we define the *Gamma function* as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Theorem 3.23. *The Gamma function satisfies the following.*

(i) $\Gamma(x+1) = x\Gamma(x)$ *for all* $x \in (0, \infty)$*;*
(ii) $\Gamma(n+1) = n!$ *for all* $n \in \mathbb{N}$*;*
(iii) $\log \Gamma$ *is convex on* $(0, \infty)$.

Theorem 3.24. *If f is a positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$ for all $x \in (0, \infty)$, $f(1) = 1$, $\log f$ is convex, then $f(x) = \Gamma(x)$ for all $x \in (0, \infty)$.*

Theorem 3.25. *We have*

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1) \cdot \ldots \cdot (x+n)}.$$

Definition 3.26. For $x > 0$ and $y > 0$, we define the *Beta function* by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

Theorem 3.27. *If $x > 0$ and $y > 0$, then*

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Example 3.28. $\Gamma(1/2) = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}$.

Theorem 3.29 (Stirling's Formula)**.** *We have*

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} = 1.$$

CHAPTER 4

The Lebesgue Integral

4.1. The Lebesgue Measure

Example 4.1. In this example, the method of the Lebesgue integral is discussed.

Definition 4.2. Let $a = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} \in \mathbb{R}^N$. We write $a \le b$ if $a_i \le b_i$ for all $1 \le i \le N$. The set $[a,b] = \{x \in \mathbb{R}^N : a \le x \le b\}$ is called a closed *interval.* The *volume* of $I = [a,b]$ is defined by

$$|I| = \prod_{i=1}^{N}(b_i - a_i).$$

Similarly we define intervals $(a,b]$, $[a,b)$, and (a,b), and their volumes are defined to be $|[a,b]|$, too.

Lemma 4.3. *If I, I_k, J_k denote intervals in $\mathbb{R}^N$, then*

(i) $I \subset J \implies |I| \le |J|$;
(ii) $I = \biguplus_{j=1}^n I_j \implies |I| = \sum_{j=1}^n |I_j|$;
(iii) $I \subset \bigcup_{j=1}^n J_j \implies |I| \le \sum_{j=1}^n |J_j|$.

Definition 4.4. We define the *outer measure* of any set $A \subset \mathbb{R}^N$ by

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} |I_j| : A \subset \bigcup_{j=1}^{\infty} I_j \text{ and } I_j \text{ are closed intervals for all } j \in \mathbb{N} \right\}.$$

Example 4.5. For $A = \{a\}$ we have $\mu^*(A) = 0$.

Lemma 4.6. *The outer measure μ^* is*

(i) *monotone:* $A \subset B \implies \mu^*(A) \le \mu^*(B)$;
(ii) *subadditive:* $\mu^*\left(\bigcup_{k=1}^\infty A_k\right) \le \sum_{k=1}^\infty \mu^*(A_k)$.

Example 4.7. Each countable set C has $\mu^*(C) = 0$.

Lemma 4.8. *If I is an interval, then $\mu^*(I) = |I|$.*

Lemma 4.9. (i) *If $d(A,B) > 0$, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$;*
(ii) *if $A_k \subset I_k$ and $\{I_k^o\}_{k\in\mathbb{N}}$ are pairwise disjoint, then $\mu^*\left(\bigcup_{k=1}^\infty A_k\right) = \sum_{k=1}^\infty \mu^*(A_k)$.*

Theorem 4.10. *We have*

$$\forall A \subset \mathbb{R}^N \; \forall \varepsilon > 0 \; \exists O \supset A : \; O \text{ open and } \mu^*(O) < \mu^*(A) + \varepsilon.$$

Definition 4.11. A set $A \subset \mathbb{R}^N$ is called *Lebesgue measurable* (or L-measurable) if

$$\forall \varepsilon > 0 \; \exists O \supset A : \; O \text{ open and } \mu^*(O \setminus A) \leq \varepsilon.$$

If A is L-measurable, then $\mu(A) = \mu^*(A)$ is called its *L-measure*.

Theorem 4.12. *The countable union of L-measurable sets is L-measurable.*

Theorem 4.13 (Examples of L-measurable Sets)**.** *Open, compact, closed, and sets with outer measure zero are L-measurable.*

Theorem 4.14. *If A is L-measurable, then $B = A^c$ can be written as*

$$B = N \uplus \bigcup_{k=1}^{\infty} F_k,$$

where $\mu^*(N) = 0$ and F_k are closed for all $k \in \mathbb{N}$.

Definition 4.15. For a set A we define the *power set* $\mathcal{P}(A)$ by $\mathcal{P}(A) = \{B : B \subset A\}$.

Definition 4.16. $\mathcal{A} \subset \mathcal{P}(X)$ is called a *σ-algebra* in X provided

(i) $X \in \mathcal{A}$;
(ii) $A \in \mathcal{A} \implies A^c \in \mathcal{A}$;
(iii) $A_k \in \mathcal{A} \; \forall k \in \mathbb{N} \implies \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$.

Theorem 4.17. *The collection of all L-measurable subsets of $\mathbb{R}^N$ is a σ-algebra.*

Theorem 4.18. *$A \subset \mathbb{R}^N$ is L-measurable iff for all $\varepsilon > 0$ there is a closed set $F \subset A$ with $\mu^*(A \setminus F) < \varepsilon$.*

Definition 4.19. A triple $(X, \mathcal{A}, \mu)$ is called a *measure space* if

(i) $X \neq \emptyset$;
(ii) $\mathcal{A}$ is a σ-algebra on X;
(iii) μ is a nonnegative and σ-additive function on $\mathcal{A}$ with $\mu(\emptyset) = 0$.

The space is called *complete* if each subset $B \subset A$ with $\mu(A) = 0$ satisfies $\mu(B) = 0$.

Theorem 4.20. *$(\mathbb{R}^N, \mathcal{A}, \mu)$ is a complete measure space, where $\mathcal{A}$ is the set of all L-measurable sets, and μ is the L-measure.*

4.2. Measurable Functions

Throughout this section, $(X, \mathcal{A}, \mu)$ is a measure space.

Definition 4.21. A function $f : X \to \overline{\mathbb{R}}$ is called *measurable* if for each $a \in \mathbb{R}$,

$$X(f \leq a) := \{x \in X : \; f(x) \leq a\}$$

is measurable (i.e., is an element of $\mathcal{A}$).

Example 4.22 (Examples of Measurable Functions)**.** (i) Constant functions are measurable.
(ii) Characteristic functions K_E are measurable iff E is measurable.
(iii) If $X = \mathbb{R}$ and $\mu = \mu_L$, then continuous and monotone functions are measurable.

Theorem 4.23. *If f and g are measurable, then*

$$X(f < g), \quad X(f \leq g), \quad \text{and} \quad X(f = g)$$

are measurable.

Theorem 4.24. *If $c \in \mathbb{R}$, f and g are measurable, then*

$$cf, \quad f+g, \quad fg, \quad |f|, \quad f^+ = \sup(f,0), \quad \text{and} \quad f^- = \sup(-f,0)$$

are measurable.

Theorem 4.25. *If $f_n : X \to \overline{\mathbb{R}}$ are measurable for all $n \in \mathbb{N}$, then*

$$\sup_{n\in\mathbb{N}} f_n, \quad \inf_{n\in\mathbb{N}} f_n, \quad \limsup_{n\to\infty} f_n, \quad \text{and} \quad \liminf_{n\to\infty} f_n$$

are measurable.

Definition 4.26. We say that two functions $f, g : X \to \overline{\mathbb{R}}$ are equal *almost everywhere* and write $f \sim g$, if there exists $N \in \mathcal{A}$ with

$$\mu(N) = 0 \quad \text{and} \quad \{x : \ f(x) \neq g(x)\} \subset N.$$

Theorem 4.27. *If $f, g : X \to \overline{\mathbb{R}}$, f is measurable, and $f \sim g$, then g is measurable, too, provided the measure space is complete.*

4.3. Summable Functions

Definition 4.28. A measurable function $f : X \to [0,\infty]$ is called *summable.*

Notation 4.29. For a summable function f, we introduce the following notation:

$$A_0(f) = \{x \in X : \ f(x) = 0\}, \quad A_\infty(f) = \{x \in X : \ f(x) = \infty\},$$

$$A_{nk}(f) = \left\{x \in X : \ \frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n}\right\},$$

$$s_n(f) = \sum_{k=1}^{\infty} \frac{k-1}{2^n}\mu\left(A_{nk}(f)\right) + \infty\mu\left(A_\infty(f)\right).$$

Theorem 4.30. *Let f and g be summable. Then for all $n \in \mathbb{N}$,*

(i) $s_n(f) \leq s_{n+1}(f)$;
(ii) $f \leq g \implies s_n(f) \leq s_n(g)$;
(iii) $f \sim g \implies s_n(f) = s_n(g)$.

Definition 4.31. If f is summable, then we define

$$\int_X f d\mu = \lim_{n\to\infty} s_n(f).$$

If B is measurable, then we define

$$\int_B f d\mu = \int_X f K_B d\mu.$$

Theorem 4.32. *If f and g are summable with $f \leq g$, then*

$$\int_X f d\mu \leq \int_X g d\mu.$$

Theorem 4.33. *If f and g are summable with $f \sim g$, then*

$$\int_X f d\mu = \int_X g d\mu.$$

Example 4.34. Let $c \geq 0$ and A be measurable. Then

$$\int_A c d\mu = c\mu(A).$$

Theorem 4.35. *If f is summable, then*

$$f \sim 0 \iff \int_X f d\mu = 0.$$

Theorem 4.36. *Let $(X, \mathcal{A}, \mu)$ be a measure space and f be summable on X. Define*

$$\nu(A) = \int_A f d\mu.$$

Then $(X, \mathcal{A}, \nu)$ is a measure space.

Theorem 4.37. *If f is summable and $c \geq 0$, then*

$$\int_X (cf) d\mu = c \int_X f d\mu.$$

Theorem 4.38 (Beppo Levi; Monotone Convergence Theorem)**.** *Let f and f_n be summable for all $n \in \mathbb{N}$ such that f_n is monotonically increasing to f, i.e., $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $f_n(x) \to f(x)$, $n \to \infty$, for all $x \in X$. Then*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Theorem 4.39. *If f and g are summable, then*

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Theorem 4.40 (Fatou Lemma)**.** *If f_n are summable for all $n \in \mathbb{N}$, then*

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu.$$

4.4. Integrable Functions

If $f : X \to \overline{\mathbb{R}}$ is measurable, then f^+ and f^- are measurable by Theorem 4.24 and hence summable. If $\int_X |f| d\mu < \infty$, then because of $f^+ \leq |f|$, $f^- \leq |f|$, and monotonicity, we have $\int_X f^+ d\mu < \infty$ and $\int_X f^- d\mu < \infty$.

Definition 4.41. A measurable function $f : X \to \overline{\mathbb{R}}$ is called *integrable* if $\int_X |f| d\mu < \infty$, we write $f \in \mathrm{L}(X)$, and then we define

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$$

and

$$\int_A f d\mu = \int_X f K_A d\mu \quad \text{if } fK_A \text{ is integrable.}$$

Lemma 4.42. (i) *If f is integrable, then $\mu(A_\infty(|f|)) = 0$;*
(ii) *if f is integrable and $f \sim g$, then so is g (in a complete measure space);*
(iii) *if f and g are integrable and $f \sim g$, then $\int_X f d\mu = \int_X g d\mu$;*
(iv) *if $f : X \to \mathbb{R}$ is integrable and $f = g - h$ such that $g, h \geq 0$ are integrable, then $\int_X f d\mu = \int_X g d\mu - \int_X h d\mu$.*

Theorem 4.43. *The integral is homogeneous and linear.*

Theorem 4.44 (Lebesgue; Dominated Convergence Theorem)**.** *Suppose $f_n : X \to \overline{\mathbb{R}}$ are integrable for all $n \in \mathbb{N}$. If $f_n \to f$, $n \to \infty$ (pointwise) and if there exists an integrable function $g : X \to [0, \infty]$ with*

$$|f_n(x)| \leq g(x) \quad \textit{for all} \quad x \in X \quad \textit{and all} \quad n \in \mathbb{N},$$

then

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Theorem 4.45. *Let I be an arbitrary interval, suppose $f(x, \cdot)$ is Lebesgue integrable on I for each $x \in [a, b]$, and define the Lebesgue integral*

$$F(x) := \int_I f(x, y)dy.$$

(i) *If for each $y \in I$ the function $f(\cdot, y)$ is continuous on $[a, b]$ and if there exists $g \in \mathrm{L}(I)$ such that*

$$|f(x, y)| \leq g(y) \quad \textit{for all} \quad x \in [a, b] \quad \textit{and all} \quad y \in I,$$

then F is continuous on $[a, b]$.

(ii) *If for each $y \in I$ the function $f(\cdot, y)$ is differentiable with respect to x and if there exists $g \in \mathrm{L}(I)$ such that*

$$\left|\frac{\partial f(x, y)}{\partial x}\right| \leq g(y) \quad \textit{for all} \quad x \in [a, b] \quad \textit{and all} \quad y \in I,$$

then F is differentiable on $[a, b]$, and we have the formula

$$F'(x) = \int_I \frac{\partial f(x, y)}{\partial x} dy.$$

Theorem 4.46. *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then it is also Lebesgue integrable, and the two integrals are the same.*

Theorem 4.47 (Arzelà)**.** *If $f_n \in \mathrm{R}[a, b]$ converge pointwise to $f \in \mathrm{R}[a, b]$ and are uniformly bounded, i.e.,*

$$|f_n(x)| \leq M \quad \textit{for all} \quad x \in [a, b] \quad \textit{and all} \quad n \in \mathbb{N},$$

then the Riemann integral satisfies

$$\lim_{n\to\infty} \int_a^b f_n dx = \int_a^b f dx.$$

4.5. The Spaces L^p

Definition 4.48. Let $p \geq 1$. Let I be an interval. We define the space $L^p(I)$ as the set of all measurable functions with the property $|f|^p \in L(I)$.

Theorem 4.49. *$L^p(I)$ is a linear space.*

Theorem 4.50 (Hölder)**.** *Suppose $p, q > 1$ satisfy $1/p + 1/q = 1$. Let $f \in L^p(I)$ and $g \in L^q(I)$. Then $fg \in L^1(I)$ and*

$$\left|\int_I fg dx\right| \leq \left(\int_I |f|^p dx\right)^{1/p} \left(\int_I |g|^q dx\right)^{1/q}.$$

Theorem 4.51 (Minkowski)**.** *For $f, g \in L^p(I)$ we have*

$$\left(\int_I |f+g|^p dx\right)^{1/p} \le \left(\int_I |f|^p dx\right)^{1/p} + \left(\int_I |g|^p dx\right)^{1/p}.$$

Definition 4.52. For each $f \in L^p(I)$ we define

$$\|f\|_p := \left(\int_I |f|^p dx\right)^{1/p}.$$

Lemma 4.53. *Suppose $g_k \in L^p(I)$ for all $k \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} \|g_n\|_p$ converges. Then $\sum_{n=1}^{\infty} g_n$ converges to a function $s \in L^p(I)$ in the L^p-sense.*

Theorem 4.54. *$L^p(I)$ is a Banach space.*

Remark 4.55. In this remark, some connections to probability theory are discussed.

Definition 4.56. We say that a sequence of complex functions $\{\phi_n\}$ is an *orthonormal* set of functions on I

$$\int_I \phi_n \overline{\phi_m} dx = \begin{cases} 0 & \text{if} \quad n \ne m \\ 1 & \text{if} \quad n = m. \end{cases}$$

If $f \in L^2(I)$ and if

$$c_n = \int_I f \overline{\phi_n} dx \quad \text{for} \quad n \in \mathbb{N},$$

then we write

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n.$$

Theorem 4.57 (Riesz–Fischer)**.** *Let $\{\phi_n\}$ be orthonormal on I. Suppose $\sum_{n=1}^{\infty} |c_n|^2$ converges and put $s_n = \sum_{k=1}^{n} c_k \phi_k$. Then there exists a function $f \in L^2(I)$ such that $\{s_n\}$ converges to f in the L^2-sense, and*

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n.$$

Definition 4.58. An orthonormal set $\{\phi_n\}$ is said to be *complete* if, for $f \in L^2(I)$, the equations

$$\int_I f \overline{\phi_n} dx = 0 \quad \text{for all} \quad n \in \mathbb{N}$$

imply that $\|f\| = 0$.

Theorem 4.59 (Parseval)**.** *Let $\{\phi_n\}$ be a complete orthonormal set. If $f \in L^2(I)$ and if*

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

then

$$\int_I |f|^2 dx = \sum_{n=1}^{\infty} |c_n|^2.$$

4.6. Signed Measures