

62. Let V be a complex vector space. An inner product on V is a mapping $(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$ such that for all $x, y, z \in V$ and all $\lambda \in \mathbb{C}$: $(x, y) = \overline{(y, x)}$, $(\lambda x, y) = \lambda(x, y)$, $(x + y, z) = (x, z) + (y, z)$, and $(x, x) > 0$ if $x \neq 0$. Also, we put $\|x\| = \sqrt{(x, x)}$.
- Prove $(x, y + z) = (x, y) + (x, z)$ and $(x, \lambda y) = \overline{\lambda}(x, y)$.
 - Prove $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$ and $\|\lambda x\| = |\lambda|\|x\|$.
 - Show that $(x, y) = x^T \overline{y}$ is an inner product on \mathbb{C}^n .
 - Show that $(f, g) = \int_a^b f(x) \overline{g(x)} dx$ is an inner product on the vector space of all continuous complex-valued functions on $[a, b]$.
63. Let V be a real vector space with inner product (\cdot, \cdot) . We call $x, y \in V$ orthogonal (write $x \perp y$) if $(x, y) = 0$. Also, we put $\|x\| = \sqrt{(x, x)}$. Prove the following statements.
- $(x - y) \perp (x + y)$ iff $\|x\| = \|y\|$.
 - $(x - z) \perp (y - z)$ iff $\|x - z\|^2 + \|y - z\|^2 = \|x - y\|^2$.
 - $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.
 - $|(x, y)| \leq \|x\|\|y\|$.
 - $\|x + y\| \leq \|x\| + \|y\|$.
64. Let V be a real vector space with inner product (\cdot, \cdot) . Let $\{e_i : i \in \mathbb{N}\} \subset V$ be orthonormal, i.e., (e_i, e_j) is zero if $i \neq j$ and is one if $i = j$. Let $n \in \mathbb{N}$, $x \in V$, and $\lambda_i \in \mathbb{R}$ for $i \in \mathbb{N}$. Prove:
- $\|\sum_{i=1}^n \lambda_i e_i\|^2 = \sum_{i=1}^n |\lambda_i|^2$;
 - $\|x - \sum_{i=1}^n \lambda_i e_i\|^2 = \|x\|^2 + \sum_{i=1}^n |\lambda_i - c_i|^2 - \sum_{i=1}^n |c_i|^2$ with $c_i = (x, e_i)$;
 - $\lim_{n \rightarrow \infty} \sum_{i=1}^n |(x, e_i)|^2$ exists and is less than or equal to $\|x\|^2$.
65. For this problem, use the inner products defined earlier for the various cases, respectively.
- Find a set of three orthonormal vectors in \mathbb{R}^3 .
 - Find α such that $e_n(t) = \alpha e^{int}$, $n \in \mathbb{Z}$ are orthonormal on $[-\pi, \pi]$.
 - For the set of real-valued polynomials on $[-1, 1]$, show that $p(x) = x$ is orthogonal to every constant function. Next, find a quadratic polynomial that is orthogonal to both p and the constant functions. Finally, find a cubic polynomial that is orthogonal to all quadratic polynomials. Hence construct an orthonormal set with three vectors.
66. For which conditions on $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are $f(b) = \alpha f(a) + \beta f'(a)$, $f'(b) = \gamma f(a) + \delta f'(a)$ symmetric boundary conditions?
67. Consider an eigenvalue problem $f'' + \lambda f = 0$ with symmetric boundary conditions.
- Show that if $f(b)f'(b) - f(a)f'(a) \leq 0$ for all $f : [a, b] \rightarrow \mathbb{R}$ satisfying the boundary conditions, then there is no negative eigenvalue.
 - Show that the condition in (a) is satisfied for Dirichlet, Neumann, and periodic boundary conditions. In which cases is it satisfied for Robin conditions?