34. If f and g are two functions of a real variable, then we define the convolution of f and g by $(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y$ provided the infinite integral exists. Show the following (provided the occuring integrals exist):

   (a) $f * g = g * f$;

   (b) $(f * g) * h = f * (g * h)$;

   (c) $(f * g)' = f * g' = f' * g$;

   (d) Find $\lim\limits_{\varepsilon \to 0^+} (\varphi_\varepsilon * f)$ where f is given by

      i. $f(x) = x$;

      ii. $f(x) = \mathrm{e}^x$

   and φ_ε is given by $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ with $\int_{\mathbb{R}} \varphi(x)\mathrm{d}x = 1$ and such that

      i. φ is a symmetric (with respect to the y-axis) nonnegative "triangle";

      ii. φ is a constant times e^{-x^2}

   (i.e., combine each of the f with each of the φ and hence solve four similar problems).

35. Solve the diffusion equation with the initial condition

   (a) $\phi(x) = \alpha$ for all $x \in \mathbb{R}$ (where $\alpha \in \mathbb{R}$);

   (b) $\phi(x) = 1$ if $|x| < l$ and zero otherwise (where $l > 0$);

   (c) $\phi(x) = 1$ for positive x and $\phi(x) = 3$ for negative x;

   (d) $\phi(x) = \mathrm{e}^{-x}$ for positive x and $\phi(x) = 0$ for negative x.

36. Let u be a solution of the diffusion equation together with $u(x,0) = \phi(x)$. Prove:

   (a) If ϕ is odd, then u is odd;

   (b) If ϕ is even, then u is even.

37. Solve the IVP $u_t - ku_{xx} + bu = 0$, $u(x,0) = \phi(x)$ (where $b > 0$) by performing a change of variables $u(x,t) = \mathrm{e}^{-bt}v(x,t)$.

38. Solve the IVP $u_t - ku_{xx} + bt^2u = 0$, $u(x,0) = \phi(x)$ (where $b > 0$) by performing a change of variables $u(x,t) = \mathrm{e}^{-bt^3/3}v(x,t)$.

39. Solve the IVP $u_t - ku_{xx} + bu_x = 0$, $u(x,0) = \phi(x)$ (where $b > 0$) by substituting $y = x - bt$.

40. Read Chapter 3 of the book. Work on at least one problem from each of its sections.