41. Find the solution of the wave equation with Dirichlet conditions (see Theorem 4.1) and

    (a) $\phi(x) = 3\sin\frac{\pi x}{l}$, $\psi(x) = 0$;

    (b) $\phi(x) = 3\sin\frac{\pi x}{l} - 2\sin\frac{3\pi x}{l}$, $\psi(x) = 4\sin\frac{\pi x}{l} + 2\sin\frac{4\pi x}{l}$.

42. Find the solution of the diffusion equation with Dirichlet conditions (see Theorem 4.2) and

    (a) $\phi(x) = 3\sin\frac{\pi x}{l}$;

    (b) $\phi(x) = 3\sin\frac{\pi x}{l} - 2\sin\frac{3\pi x}{l}$.

43. Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature one everywhere. Suddenly both ends are plunged into a bath of temperature zero. Write the differential equation, boundary conditions, and initial conditions. Write the formula for the temperature $u(x,t)$ at later times. In this problem, you can use the infinite series expansion $\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\frac{(2n-1)\pi x}{l} = \frac{\pi}{4}$.

44. Find all eigenvalues and eigenfunctions of $f'' + \lambda f = 0$, $f(0) = f(\pi) = 0$. How many zeros inside the interval $(0,\pi)$ does the nth eigenfunction of the problem have?

45. Find all eigenvalues and eigenfunctions of $f'' + \lambda f = 0$, $f(-\pi) = f(\pi)$, $f'(-\pi) = f'(\pi)$. Also show that the eigenfunctions are orthogonal in the sense that $\int_{-\pi}^{\pi} e_1(x)e_2(x)\mathrm{d}x = 0$ whenever e_1 and e_2 are eigenfunctions corresponding to two different eigenvalues.

46. Consider the second order difference equation $\Delta^2 f_k + \lambda f_{k+1} = 0$ (where the forward difference operator Δ is defined by $\Delta f_k = f_{k+1} - f_k$). Determine the values of λ (and of θ) for which $f_k = \cos(k\theta)$, $f_k = \sin(k\theta)$, $f_k = \cosh(k\theta)$, and $f_k = \sinh(k\theta)$ solve the equation and hence find the general solution. Then find the eigenvalues and eigenfunctions of $\Delta^2 f_k + \lambda f_{k+1} = 0$, $f_0 = f_N = 0$, where $N \in \mathbb{N}$. How many eigenvalues does this problem have?

47. Separate the variables for the equation $tu_t = u_{xx} + 2u$ with $u(0,t) = u(\pi,t) = 0$. Show that the solution of this problem satisfying in addition $u(x,0) = 0$ is not unique.

48. Use the method of separation of variables to find solutions (as many as possible) of the following problems:

    (a) $u_{xx} + u_{tt} = 0$ ($0 < x < a$, $t > 0$), $u(0,t) = u(a,t) = 0$;

    (b) $u_{xx} + u_{tt} = 0$ ($0 < x < a$, $t > 0$), $u_x(0,t) = u_x(a,t) = 0$;

    (c) $u_{tt} + a^2 u_{xxxx} = 0$ ($0 < x < l$, $t > 0$), $u(0,t) = u(l,t) = u_{xx}(0,t) = u_{xx}(l,t) = 0$.