Missouri University of Science and Technology

Section 7.2

The Definition of the Laplace Transform

Laplace Transforms: The Basic Goal

We wish to solve IVPs involving DEs of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t)$$

with initial conditions

$$y(0) = y_0, y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}$$

If $g(t)$ is reasonably straightforward, we can use undetermined coefficients or variation of parameters.

If $g(t)$ is more complicated, Laplace transforms often work.

Solving IVPs Using Laplace: The Basic Idea

1. Convert the initial value problem into an algebraic equation using the Laplace transform
2. Solve the algebraic equation
3. Convert the solution of the algebraic equation into a solution of the initial value problem using the inverse Laplace transform

Are Laplace Transforms Useful?

Yes!

The Laplace transform allows us to solve initial value problems that cannot be solved using other techniques.

The Laplace transform is a fundamental tool in systems and in control theory.

The Laplace Transform

Let f be a function defined on $[0, \infty)$. Then, the Laplace transform of f at s is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

for those values of s for which the improper integral converges.

Example 1

Find the Laplace transform of $f(t) = e^{4t}$.

Example 2

Find the Laplace transform of $f(t) = 1$.

Property

The Laplace transform is a linear transform.

Thus, we can write

$$\begin{aligned}\mathcal{L}\{af(t) + bg(t)\} &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \\ &= aF(s) + bG(s)\end{aligned}$$

Fact

$$\mathcal{L}\{e^{ibt}\} = \frac{1}{s - ib}, s > \mathrm{Re}(ib) = 0$$

Example 3

Find the Laplace transforms of the functions $f(t) = \cos(bt)$ and $g(t) = \sin(bt)$ where b is a real constant.

Table of Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$t^n, n = 1,2, \ldots$	$\frac{n!}{s^{n+1}}, s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, s > 0$

Example 4

Find the Laplace transform of

$$f(t) = 2e^{3t} + 3t^2 + 9\sin 4t$$

Piecewise Continuity

If a function f is continuous at every point of a finite interval $[a,b]$ except possibly at a finite number of points where f has a jump discontinuity, then f is said to be piecewise continuous on $[a,b]$.

If f is piecewise continuous on every possible interval $[0,N]$ for $N > 0$, then f is said to be piecewise continuous on $[0,\infty)$.

Exponential Order

A function f is said to be of exponential order α if there exist positive constants T and M such that

$$|f(t)| \leq Me^{\alpha t}$$

for all $t \geq T$.

Theorem

If f is piecewise continuous on $[0,\infty)$ and of exponential order α, then $\mathcal{L}\{f(t)\}$ exists for $s > \alpha$.

Example 5

Find the Laplace transform of

$$f(t) = \begin{cases} e^t & 0 \le t < 1 \\ 1 + e^{2t} & 1 < t \end{cases}$$