Missouri University of Science and Technology — Founded 1870 | Rolla, Missouri

Section 7.3

Properties of the Laplace Transform

Goals for Today

Last time, we considered how to find Laplace transforms of a few specific (and simple) types of functions and their sums

Today, we need to consider how to transform derivatives.

We also need to consider how to find Laplace transforms of more complicated functions.

Review: The Laplace Transform

Let f be a function defined on $[0, \infty)$. Then, the Laplace transform of f at s is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

for those values of s for which the improper integral converges.

Laplace Transforms of Derivatives

Let f be continuous on $[0, \infty)$ and let f' be piecewise continuous on $[0, \infty)$ with both of exponential order α. Then, for $s > \alpha$,

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) \\ &= sF(s) - f(0) \end{aligned}$$

Note: Future results on the Laplace transform will always include similar assumptions. We will not typically bother to mention these assumptions.

Example 1

Prove the preceding theorem.

Example 2

Use the preceding theorem to find

$$\mathcal{L}\{f''(t)\}$$

Laplace Transforms of Derivatives

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$$

Table of Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$t^n, n = 1,2, \ldots$	$\frac{n!}{s^{n+1}}, s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, s > 0$

Translation Theorem for Laplace Transforms

If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Example 3

Find $\mathcal{L}\{e^{at}\cos bt\}$.

Theorem

$$\mathcal{L}\{tf(t)\} = (-1)\frac{d}{ds}F(s)$$

Example 4

Find $\mathcal{L}\{t\cos bt\}$

Theorem

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s), \qquad n = 1,2,\ldots$$

Example 5

Find $\mathcal{L}\{t^2 \cos bt\}$