

Impulse

The integral

$$I_g(\tau) = \int_{-\tau}^{\tau} g(t)dt$$

is called the impulse of the force $\overset{\circ}{g}$ over the time interval $-\tau < t < \tau$.

Example 1

Consider the force defined by

$$g_n(t) = \begin{cases} \frac{n}{2} & -\frac{1}{n} < t < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

where n is a positive integer.

- a) Sketch $g_n(t)$ for n = 1, 2, 4, and 8.
- b) Calculate the impulse of $g_n(t)$ over the time interval $-\frac{1}{n} < t < \frac{1}{n}.$

The Dirac Delta

For each bounded continuous function
$$f$$
 on $(-\infty,\infty)$, define
$$\int\limits_{-\infty}^{\infty}f(t)\delta(t)dt=\lim_{n\to\infty}\int\limits_{-\infty}^{\infty}f(t)g_n(t)dt$$

 δ is called the Dirac delta or the unit impulse. Note that δ is not a function.

The Dirac Delta – Rewriting the Definition

Since
$$g_n(t)$$
 is zero outside $-\frac{1}{n} < t < \frac{1}{n'}$ we can rewrite as follows:
$$\int\limits_{-\infty}^{\infty} f(t) \delta(t) dt = \lim_{n \to \infty} \int\limits_{-\infty}^{\infty} f(t) g_n(t) dt$$

$$= \lim_{n \to \infty} \int\limits_{-1/n}^{1/n} f(t) g_n(t) dt$$

$$= \lim_{n \to \infty} \int\limits_{-1/n}^{1/n} f(t) \frac{n}{2} dt$$

$$=\lim_{n\to\infty}\frac{1}{\frac{2}{n}}\int_{-1/n}^{1/n}f(t)dt$$

This limit computes the average value of f(t) over the interval $-\frac{1}{n} < t < \frac{1}{n}$ as that interval becomes increasingly small. Thus, this limit converges to f(0).

The Sifting Property of the Dirac Delta

For each bounded continuous function f on $(-\infty, \infty)$, define

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \lim_{n \to \infty} \int_{-\infty}^{\infty} f(t)g_n(t)dt = \lim_{n \to \infty} \frac{1}{n} \int_{-1/n}^{1/n} f(t)dt = f(0)$$

This is the sifting property of the Dirac delta.

If we translate the Dirac delta horizontally by a, we obtain the more generalized version of the sifting property

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

The Laplace Transform of the Dirac Delta

Recall that the Laplace transform is defined as

$$\mathcal{L}{f(t)} = \int_{0}^{\infty} e^{-st} f(t) dt$$

Thus, the Laplace transform of the Dirac delta is

$$\mathcal{L}\{\delta(t-a)\} = \int\limits_{-\infty}^{\infty} e^{-st} \delta(t-a) dt = e^{-as}$$

Similarly,

$$\mathcal{L}\{f(t)\delta(t-a)\} = \int_{0}^{\infty} e^{-st} f(t)\delta(t-a)dt = f(a)e^{-as}$$

