Missouri University of Science and Technology

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## Sections 9.1 and 5.1

Introduction to Systems of Linear Differential Equations

Interconnected Fluid Tanks

## Goals for Today

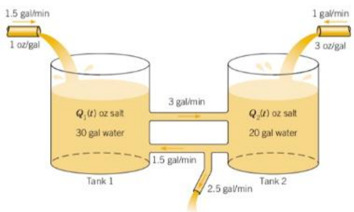
Today, we will consider one application of systems of linear differential equations: interconnected fluid tanks.

## Example 1

Consider an interconnected system of two tanks, Tank 1 and Tank 2. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows from outside the system into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows from outside the system into Tank 2 at a rate of 1 gal/min. The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min while the remainder leaves the system.

Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time $t$. Set up a system, including initial conditions, which models the flow process.

## Example 1




## Systems of Linear Differential Equations

If a system of differential equations can be expressed in the form

$$\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\\
&\vdots\\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

it is said to be a first-order linear system in normal form.

If $f_1(t) = f_2(t) = \cdots = f_n(t) = 0$, the system is homogeneous.

## Systems in Matrix Form

In matrix form, a first-order linear system can be written as

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$$

where $A$ is the $n \times n$ coefficient matrix $A = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$

and $\mathbf{f}(t) = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$.

$\mathbf{x}(t)$ is the solution vector $\mathbf{x}(t) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

## Example 2

Express the given system of differential equations in matrix form.

$$x' = x + y + z$$
$$y' = 2z - x$$
$$z' = 4y$$

## Converting Higher-Order Equations to Systems

To rewrite an $n^{\text{th}}$-order linear homogeneous differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

as a first-order linear system, let

$$x_1 = y$$
$$x_2 = y'$$
$$\vdots$$
$$x_n = y^{(n-1)}$$

Then the system can be expressed as

$$x_1' = x_2$$
$$x_2' = x_3$$
$$\vdots$$
$$x_{n-1}' = x_n$$
$$x_n' = -\frac{a_0}{a_n} x_1 - \frac{a_1}{a_n} x_2 - \cdots - \frac{a_{n-1}}{a_n} x_n$$

## Example 3

Express the damped, unforced spring-mass equation

$$my'' + \gamma y' + ky = 0$$

as a matrix system in normal form.

## Example 4

Express the higher-order system

$$x'' + 3x' - y' + 2y = 0$$
$$y'' + x' + 3y' + y = 0$$

as a matrix system in normal form.