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Section 9.3

Matrices and Vectors

Goals for Today

In this chapter, we will look at solving systems of linear differential equations.

To prepare, we need to explore some basic linear algebra.

Matrices

A matrix is a rectangular array of numbers (or functions).

The plural of matrix is matrices.

An $m \times n$ matrix has m rows and n columns.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a 3×2 matrix.

Vectors

A matrix with only one column is called a (column) vector.

In printed materials, vectors are usually denoted by lowercase boldface letters.

Example: $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

In handwritten materials, vectors are usually denoted with an arrow above the lowercase letter.

Example: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Standard Notation for Matrices

If A is an $m \times n$ matrix, then

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$$

where $\mathbf{a}_j$ is the vector corresponding to the jth column of A; and

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

where a_{ij} represents the entry in row i, column j.

The Zero Matrix

An $m \times n$ matrix whose entries are all zero is called a zero matrix, denoted 0.

The size of the matrix 0 should be clear from the context, and it should also be clear from the context whether 0 refers to a scalar or a matrix.

Identity Matrices

2×2 identity matrix: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3×3 identity matrix: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$n \times n$ identity matrix: $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

When the size is clear from the context, we often simply refer to the identity matrix as I.

Equality of Matrices

Two matrices are equal if they have the same size and all corresponding entries are equal.

Sums of Matrices

To add two matrices of the same size, simply add all corresponding entries.

Matrices of different sizes cannot be added together.

Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then

$$cA = [c(a_{ij})]$$

In other words, we simply multiply every entry of A by the scalar c.

Example 1

Let $A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 & 2 \\ 0 & -4 & -1 \end{bmatrix}$.

Calculate $A - 2B$.

Properties of Matrix Arithmetic

Let A, B, and C be matrices of the same size and let r and s be scalars. Then,

1. $A + B = B + A$
2. $A + B + C = (A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

Matrix Multiplication

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix where the entry in row i and column j of AB is calculated as the dot product of row i of A and column j of B.

Example 2

Calculate each product, if possible.

$$\begin{bmatrix} 1 & 4 & 2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 0 & -4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -5 & 1 \\ 3 & 4 & -6 \end{bmatrix}$$

Example 3

Let $A = \begin{bmatrix} 1 & -2 & 5 \\ -3 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 1 & 7 \\ 2 & 1 \end{bmatrix}$.

Calculate AB and BA.

Example 4

Let $A = \begin{bmatrix} -5 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -5 & -3 \end{bmatrix}$.
Calculate AB and BA.

Warnings About Matrix Multiplication

In general, $AB \neq BA$.

The cancellation laws do not hold for matrix multiplication. Thus, if $AB = AC$, then it is not true in general that $B = C$.

If $AB = 0$, we cannot conclude in general that either $A = 0$ or $B = 0$.

Properties of Matrix Multiplication

Let A, B, and C be matrices whose sizes are appropriate for the following products and let r be a scalar. Then,

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $r(AB) = (rA)B = A(rB)$
5. $I_m A = A = AI_n$ if A is $m \times n$

The Transpose of a Matrix

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix A^T whose columns are the corresponding rows of A.

Symbolically, if $A = [a_{ij}]$, then $A^T = [a_{ji}]$.

Example 5

Given $A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 2 & 0 \end{bmatrix}$, calculate A^T.

The Inverse of a Matrix

An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I_n \text{ and } AC = I_n$$

C is the inverse of A and we usually write

$$C = A^{-1}$$

An invertible matrix is called nonsingular.

A noninvertible matrix is called singular.

Invertibility of Matrices

Let A be an $n \times n$ matrix.

If $\det A \neq 0$, then A is invertible.

If $\det A = 0$, then A is not invertible.

Non-square matrices are not invertible.

The Inverse of a 2×2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then,

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 6

Calculate the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Differentiation and Integration of Matrices

Differentiation and integration of matrices are performed entry-by-entry.

Example 7

Consider the matrix

$$A = \begin{bmatrix} \cos t & e^{-t} \\ 1 & \sin t \end{bmatrix}$$

a) Calculate $\frac{d}{dt}A$.

b) Calculate $\int_0^1 A\, dt$.