Section 9.4

Linear Systems in Normal Form

Goals for Today

In this section, we will explore various properties of solutions to linear systems in normal form.

In the next section, we will actually solve these systems.

Systems in Normal Form

A linear system of n differential equations is in normal form if it is expressed as

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

where $\mathbf{x}(t)$ is $n \times 1$, $A(t)$ is $n \times n$, and $\mathbf{f}(t)$ is $n \times 1$.

If $\mathbf{f}(t) = \mathbf{0}$, the system is homogeneous.

If all elements of $A(t)$ are constant, the system is said to have constant coefficients.

Existence of Unique Solutions to Linear Systems

If $A(t)$ and $\mathbf{f}(t)$ are continuous on an open interval I which contains t_0, then the initial value problem

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution $\mathbf{x}(t)$ on the interval I for any choice of $\mathbf{x}_0$.

Solutions of Homogeneous Linear Systems

If $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are solutions to the homogeneous linear system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

then $c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k$ is also a solution for any constants $c_1, \ldots, c_k$.

The Wronskian of Vector Functions

The Wronskian of n vector functions

$$\mathbf{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \ldots, \mathbf{x}_n(t) = \begin{bmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$$

is defined as the function

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix}$$

Provided that $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t_0) \neq 0$ for at least one point t_0 in an interval I, we conclude that the functions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent on I.

Example 1

Determine whether the given vector functions are linearly independent or linearly dependent on the interval $(-\infty, \infty)$.

$$\mathbf{x}_1 = \begin{bmatrix} e^{-5t} \\ -e^{-5t} \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 6e^{2t} \\ e^{2t} \end{bmatrix}$$

Fundamental Sets of Solutions

If $\mathbf{x}_1, \dots, \mathbf{x}_n$ are a linearly independent set of solutions of the homogeneous linear system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

where $A(t)$ is $n \times n$, then $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is a fundamental set of solutions for the homogeneous linear system.

General Solutions of Homogeneous Linear Systems

Let $A(t)$ be an $n \times n$ matrix function which is continuous on an interval I. If the homogeneous linear system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

has a fundamental set of solutions $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ on the interval I, then the general solution of the homogeneous system is

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

where $c_1, \dots, c_n$ are constants.

General Solutions of Nonhomogeneous Linear Systems

Let $A(t)$ be an $n \times n$ matrix function which is continuous on an interval I. If $\mathbf{x}_p$ is a particular solution to the nonhomogeneous system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

on the interval I and $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is a fundamental set of solutions for the associated homogenous equation on I, then the general solution of the nonhomogeneous system is

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$$

where $\mathbf{x}_h(t) = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$.

Example 2

Recall (from Example 1) that $\mathbf{x}_1 = \begin{bmatrix} e^{-5t} \\ -e^{-5t} \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 6e^{2t} \\ e^{2t} \end{bmatrix}$ are linearly independent.

a) Verify that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the linear system

$$\mathbf{x}' = \begin{bmatrix} 1 & 6 \\ 1 & -4 \end{bmatrix}\mathbf{x}$$

b) State the general solution of the system.

Fundamental Matrices

Let $A(t)$ be an $n \times n$ matrix function which is continuous on an interval I. If the homogeneous linear system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

has a fundamental set of solutions $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ on the interval I, then

$$X(t) = [\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_n]$$

is called a fundamental matrix for the homogeneous system.

$X(t)$ is a solution of the system $X'(t) = A(t)X(t)$.

Example 2 (continued)

$\mathbf{x}_1 = \begin{bmatrix} e^{-5t} \\ -e^{-5t} \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 6e^{2t} \\ e^{2t} \end{bmatrix}$ form a fundamental set of solutions of the linear system

$$\mathbf{x}' = \begin{bmatrix} 1 & 6 \\ 1 & -4 \end{bmatrix} \mathbf{x}$$

c) State a fundamental matrix $X(t)$ for the linear system.

d) Verify that the fundamental matrix $X(t)$ is a solution of the linear system.

e) State the general solution of the system using $X(t)$