Missouri University of Science and Technology

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Section 9.5

Homogeneous Linear Systems with Constant Coefficients

Goals for Today

In this section, we will introduce eigenvalue and eigenvectors and explore their usage in solving homogeneous linear systems of the form

$$\mathbf{x}' = A\mathbf{x}$$

where A is an $n \times n$ constant matrix.

Solving $\mathbf{x}' = A\mathbf{x}$

Assume $\mathbf{x}' = A\mathbf{x}$ with A constant has solutions of the form

$$\mathbf{x} = e^{\lambda t}\mathbf{v}$$

where $\mathbf{v}$ is a nonzero constant vector.

If we plug our assumed solution into $\mathbf{x}' = A\mathbf{x}$, we get

$$\left(e^{\lambda t}\mathbf{v}\right)' = Ae^{\lambda t}\mathbf{v}$$
$$\lambda e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\mathbf{v}$$

Since $e^{\lambda t}$ is a nonzero scalar quantity, we can divide both sides by $e^{\lambda t}$ and obtain

$$\lambda\mathbf{v} = A\mathbf{v}$$

Eigenvalues and Eigenvectors

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ.

A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{v}$ of $A\mathbf{v} = \lambda\mathbf{v}$. Such an $\mathbf{v}$ is called an eigenvector of A corresponding to λ.

Solving $\mathbf{x}' = A\mathbf{x}$

$\mathbf{x}' = A\mathbf{x}$ with A constant has solutions of the form

$$\mathbf{x} = e^{\lambda t}\mathbf{v}$$

where λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$.

Finding Eigenvalues

The eigenvalue-eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

For this equation to have a nontrivial solution, properties of linear algebra tell us that $(A - \lambda I)$ must *not* be invertible.

Recall: If $\det A \neq 0$, then A is invertible.
If $\det A = 0$, then A is not invertible.

Thus, we seek values of λ for which

$$\det(A - \lambda I) = 0$$

$\det(A - \lambda I) = 0$ is called the characteristic equation of A.

Example 1

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 \\ 1 & -4 \end{bmatrix}$$

a) Find the eigenvalues of A

Finding Eigenvectors

The eigenvalue-eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

Once we know an eigenvalue λ of A, we can plug λ into $(A - \lambda I)\mathbf{v} = \mathbf{0}$ and use row operations to solve for a corresponding eigenvector $\mathbf{v}$.

Example 1 (continued)

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 \\ 1 & -4 \end{bmatrix}$$

b) Find an eigenvector of A corresponding to $\lambda_1 = -5$.
c) Find an eigenvector of A corresponding to $\lambda_2 = 2$.
d) State two solutions to the linear system $\mathbf{x}' = A\mathbf{x}$.

Solving $\mathbf{x}' = A\mathbf{x}$ with n Linearly Independent Eigenvectors

If the $n \times n$ constant matrix A has n real eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ and the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a linearly independent set, then the general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$$

where $c_1, \dots, c_n$ are scalars.

Theorem

If $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of the matrix A with corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_m$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a linearly independent set.

Example 1 (continued)

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 \\ 1 & -4 \end{bmatrix}$$

e) State the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$.

Example 2

Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & -3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

a) Find the eigenvalues and eigenvectors of A.

b) Find a general solution of the system $\mathbf{x}' = A\mathbf{x}$.

Example 3

Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 2 & 0 & -1 \\ 4 & 4 & -4 \end{bmatrix}$$

a) Given that the characteristic equation of A is $(2-\lambda)(2+\lambda)^2 = 0$, find its eigenvalues and eigenvectors.

b) Find a general solution of the system $\mathbf{x}' = A\mathbf{x}$.