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Section 9.6

Complex Eigenvalues

Goals for Today

In this section, we will explore the solutions of homogeneous linear systems of the form

$$\mathbf{x}' = A\mathbf{x}$$

where A is an $n \times n$ constant matrix with all real entries and complex eigenvalues.

Recall: Solving $\mathbf{x}' = A\mathbf{x}$

$\mathbf{x}' = A\mathbf{x}$ with A constant has solutions of the form

$$\mathbf{x} = e^{\lambda t}\mathbf{v}$$

where λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$.

Recall: Complex Roots of the Auxiliary Equation

Consider the differential equation $ay'' + by' + cy = 0$.

If its auxiliary equation $ar^2 + br + c = 0$ has complex roots $r = \lambda \pm \mu i$, then

$$y_1 = e^{\lambda t}\cos(\mu t)$$

and

$$y_2 = e^{\lambda t}\sin(\mu t)$$

are linearly independent real solutions of the DE.

Matrices with Complex Eigenvalues

Let A be an $n \times n$ matrix whose entries are real, and let $\lambda_1 = \alpha + \beta i$ be an eigenvalue of A with corresponding eigenvector $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$.

Then, $\lambda_2 = \alpha - \beta i$ is also an eigenvalue of A with corresponding eigenvector $\mathbf{v}_2 = \mathbf{a} - i\mathbf{b}$.

Solving $\mathbf{x}' = A\mathbf{x}$ with Complex Eigenvalues

Let A be an $n \times n$ matrix whose entries are real.

If $\lambda_1 = \alpha + \beta i$ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$, then two linearly independent real solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}_1 = e^{\alpha t}\cos(\beta t)\,\mathbf{a} - e^{\alpha t}\sin(\beta t)\,\mathbf{b}$$
$$\mathbf{x}_2 = e^{\alpha t}\sin(\beta t)\,\mathbf{a} + e^{\alpha t}\cos(\beta t)\,\mathbf{b}$$

Example 1

Consider the matrix $A = \begin{bmatrix} -3 & 2 \\ -5 & -1 \end{bmatrix}$.

a) Find the eigenvalues and eigenvectors of A.
b) Find a general solution of the system $\mathbf{x}' = A\mathbf{x}$.
c) Find a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$.

Example 2

Find the solution to the given initial value problem.

$$\mathbf{x}' = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$