Section 9.7

Nonhomogeneous Linear Systems

Goals for Today

In this section, we will explore the solutions of nonhomogeneous linear systems of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$$

where A is a real $n \times n$ constant matrix and $\mathbf{f}$ is a vector function which depends only on the independent variable.

We will consider two methods:

1. Method of Undetermined Coefficients
2. Variation of Parameters

Recall: The Method of Undetermined Coefficients

When solving $ay'' + by' + cy = g(t)$, the Method of Undetermined Coefficients suggests that the form of $y_p(t)$ is a linear combination of $g(t)$ and all distinct functions generated by repeated differentiation of $g(t)$, provided that none of those functions duplicate the functions found in the homogeneous solution $y_h(t)$.

If our initial guess for y_p duplicates y_h, multiply the entire guess for y_p by the smallest positive integer power of t which ensures that no portion of y_h is duplicated.

Example 1

Consider the matrix $A = \begin{bmatrix} 2 & 4 & -2 \\ 2 & 0 & -1 \\ 4 & 4 & -4 \end{bmatrix}$.

Find a general solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where $\mathbf{f} = \begin{bmatrix} 2t \\ 1 \\ 0 \end{bmatrix}$ given that the solution of the associated homogeneous equation $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}_h = c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Recall: Variation of Parameters

Consider the differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

If $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the associated homogeneous equation, then our particular solution of the nonhomogeneous equation has the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1 = \int \frac{g(t)W_1}{W} dt \text{ and } u_2 = \int \frac{g(t)W_2}{W} dt$$

Derivation: Variation of Parameters for Linear Systems

Let $X(t)$ be a fundamental matrix for the homogeneous system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

Thus, a general solution to the homogeneous system can be stated as

$$\mathbf{x}_h(t) = X(t)\mathbf{c}$$

where $\mathbf{c}$ is an arbitrary constant vector.

Then, assume that the nonhomogeneous system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

has a particular solution of the form

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t)$$

Derivation: Variation of Parameters for Linear Systems

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

Assume

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t)$$

Substituting, we obtain

$$\big(X(t)\mathbf{u}(t)\big)' = A(t)X(t)\mathbf{u}(t) + \mathbf{f}(t)$$
$$X'(t)\mathbf{u}(t) + X(t)\mathbf{u}'(t) = A(t)X(t)\mathbf{u}(t) + \mathbf{f}(t)$$

Since $X(t)$ is a fundamental matrix, we know $X'(t) = \mathrm{A}(t)X(t)$.
Thus,

$$\mathrm{A}(t)X(t)\mathbf{u}(t) + X(t)\mathbf{u}'(t) = A(t)X(t)\mathbf{u}(t) + \mathbf{f}(t)$$
$$X(t)\mathbf{u}'(t) = \mathbf{f}(t)$$

Derivation: Variation of Parameters for Linear Systems

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

$$X(t)\mathbf{u}'(t) = \mathbf{f}(t)$$

We know $X(t)$ is invertible, so we get

$$\mathbf{u}'(t) = X^{-1}(t)\mathbf{f}(t)$$

and, thus,

$$\mathbf{u}(t) = \int X^{-1}(t)\mathbf{f}(t)dt$$

Variation of Parameters for Linear Systems

To solve $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$, first find a fundamental matrix $X(t)$ of the associated homogeneous equation $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ and note that the homogeneous solution will be $\mathbf{x}_h(t) = X(t)\mathbf{c}$.

Then, a particular solution of the nonhomogeneous equation will be

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t)$$

where

$$\mathbf{u}(t) = \int X^{-1}(t)\mathbf{f}(t)dt$$

Example 2

Consider the matrix $A = \begin{bmatrix} 1 & 6 \\ 1 & -4 \end{bmatrix}$.

Find a general solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where $\mathbf{f} = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ given that the solution of the associated homogeneous equation $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}_h = c_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$