Exam 3 Review Sheet

Math 3304

7.5-7.9 - Solving Initial Value Problems with Laplace Transforms

Problem 1. Find the solution of the initial value problem

$$y'' + 2y' + 2y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 0.$$

Then compute the value of the solution, accurate to five decimal places, when $t = 6$.

Problem 2. Solve the initial value problem

$$y'' - 4y' + 5y = \delta(t-2\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Express your final answer without any unit step functions.

Problem 3. Solve the integrodifferential equation

$$y'(t) = 1 - \int_0^t e^{-2\tau} y(t-\tau) d\tau, \quad y(0) = 1.$$

Problem 4. Find the solution of the integro-differential equation

$$y'(t) + 3\int_0^t e^{-4r} y(t-r)\, dr = 1, \quad y(0) = 0.$$

Problem 5. Solve the initial value problem

$$y'' + 4y = 1 - u(t-3\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Problem 6. Solve the initial value problem

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = -2.$$

Problem 7. Solve the integral equation

$$y(t) + \int_0^t e^{-\tau} y(t-\tau)\, d\tau = 1.$$

Problem 8. (a) Find the solution $y = y(t)$ of the initial value problem

$$y'' + 3y' + 2y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 1.$$

(b) Which is greater, $y(6)$ or $y(1)$? Justify your answer.

Problem 9. Solve the integral equation

$$y(t) = 3t^2 - e^{-t} - \int_0^t y\,(u) e^{t-u}\,du.$$

Problem 10. Find the solution of the initial value problem

$$y'' + 4y = 2 - 4u(t - \frac{\pi}{2}), \quad y(0) = 0, \quad y'(0) = 0.$$

Problem 11. Find the solution of the initial value problem

$$y'' + y = \frac{1}{\pi}\delta(t - \pi), \quad y(0) = 0, \quad y'(0) = -1.$$

Write your solution as a piecewise defined function and sketch its graph on the interval

$$0 \le t \le 3\pi.$$

Problem 12. Solve the integral equation

$$y(t) + 5\int_0^t y\,(t - r)\cos(2r)\,dr = e^{-3t}.$$

Problem 13. Compute the inverse Laplace transform of

$$F(s) = \frac{2s^2 + 6s + 12}{s(s-2)(s+2)}.$$

Problem 14. (a) Solve the initial value problem

$$y'' + 4y = g(t), \quad y(0) = 1, \quad y'(0) = 0,$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 3\pi, \\ 1, & 3\pi \le t < \infty. \end{cases}$$

(b) Which is greater, $y(2\pi)$ or $y(5\pi)$? Justify your answer.

Problem 15. Solve the initial value problem

$$y^{(4)} - y = 2\delta(t - 1), \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

Problem 16. (a) Find the solution of the initial value problem

$$y'' + y = g(t), \quad y(0) = 1, \quad y'(0) = -1,$$

where

$$g(t) = \begin{cases} 1-t, & t < 1, \\ 0, & t \geq 1. \end{cases}$$

(b) Draw the graph of the solution on the interval

$$0 \leq t \leq 3\pi.$$

Problem 17. Solve the integro-differential equation

$$y'(t) + 2y(t) - 2\int_0^t y\,(\xi)\sin(t-\xi)\,d\xi = -\sin(t),$$

subject to the initial condition

$$y(0) = 1.$$

Problem 18. (a) Solve the initial value problem

$$y'' + 4y = \delta(t-\pi), \quad y(0) = 1/2\,, \quad y'(0) = 0.$$

(b) Which is greater, $y(\pi)$ or $y(3\pi/2)$? Justify your answer.

Problem 19. Solve the initial value problem

$$y'' + 4y' + 13y = h(t), \quad y(0) = y'(0) = 0,$$

where

$$h(t) = \begin{cases} 0, & 0 \leq t < \pi, \\ 13, & \pi \leq t < \infty. \end{cases}$$

Problem 20. Find the solution $y(t)$ of the integral equation

$$y(t) - 9\int_0^t (\,t-\tau)y(\tau)\,d\tau = 9t^2.$$

Problem 21. Solve the initial value problem

$$y'' + 3y' + 2y = u(t-10), \quad y(0) = y'(0) = 0$$

Problem 22. Find the solution of the integral equation

$$y(t) - \int_0^t y\,(\tau)sin(t-\tau)d\tau = \delta(t-\pi)cos(t)$$

Problem 23. Solve the initial value problem

$$y' + 3\int_0^t e^{-4(t-\tau)}\,y(\tau)d\tau = 0, \qquad y(0) = 1$$

Problem 24. Solve the initial value problem

$$y''(t) + 2y'(t) + 5y(t) = 2\delta(t - \frac{\pi}{2})sin(t), \qquad y(0) = 1, y'(0) = -1$$

Problem 25. Solve the initial value problem

$$y'' + 9y = f(t), y'(0) = 1$$

where

$$f(t) = \begin{cases} 3 & 0 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases}$$

Problem 26. Solve the Integro-differential Equation

$$y' + \int_0^t c\,os(t - \tau)y(\tau)d\tau = 1, \quad y(0) = 0$$

Problem 27. Given that $\mathcal{L}(tsin(at)) = \frac{2as}{(s^2+a^2)^2}$, solve the IVP

$$y'' + 9y = \delta(t - 3\pi) + cos(3t), \quad y(0) = y'(0) = 0$$

Calculate the value of $y(\frac{2\pi}{3})$.

Problem 28. Find the solution of the integral equation

$$y(t) - \int_0^t \tau\, y(t - \tau)d\tau = e^t$$

Problem 29. Solve the initial value problem

$$y'' + y = u_\pi(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

Problem 30. Solve the integro-differential equation

$$2y'(t) = \int_0^t (t - \tau)^2 y(\tau)\, d\tau - 2t$$

with

$$y(0) = 1.$$

Problem 31. Solve the initial value problem

$$y'' + 3y' + 2y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t. \end{cases}$$

Problem 32. Solve the integral equation

$$y(t) + 4\int_0^t (t-\tau)y(\tau)\,d\tau = t\,u_1(t) - t.$$

Problem 33. Solve the initial value problem

$$y'' - 4y' + 5y = \delta(t-\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Problem 34. Find the solution of the integral equation

$$y(t) - \int_0^t \sin(t-\tau)y(\tau)\,d\tau = t.$$

Problem 35. Use the Laplace transform to solve the initial value problem

$$y' - y = f(t), \qquad y(0) = 1,$$

where

$$f(t) = \begin{cases} 0, & 0 \le t < 4, \\ e^{3(t-4)}, & t \ge 4. \end{cases}$$

Problem 36. Solve the integral equation

$$y(t) + 3\int_0^t y(\tau)\sin(t-\tau)\,d\tau = t.$$

Problem 37. Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t-\pi), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Problem 38. Find $g(t)$, which is the following inverse Laplace transform:

$$g(t) = \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2+4}\right\}$$

Problem 39. Make use of Laplace transform to solve the following initial value problem:

$$y'' + 3y' + 2y = 1, \quad y(0) = 0, \quad y'(0) = 0$$

Problem 40. Solve the following integral equation:

$$y(t) - \int_0^t e^{-(t-v)}\,y(v)\,dv = u(t-1)$$

Problem 41. Solve the given initial value problem using Laplace transforms:

$$y'' - y' - 2y = 0, \quad y(0) = -2, \quad y'(0) = 5$$

Problem 42. Find the solution of the integral equation:

$$y(t) - \int_0^t \sin(t-\tau)y(\tau)\,d\tau = t$$

Problem 43. Solve the initial value problem:

$$2y'' + 4y' + 10y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 0$$

Problem 44. Use Laplace Transform method to solve the initial value problem:

$$y'' + 4y = 2; \quad y(0) = 0, \quad y'(0) = 0$$

Problem 45. Solve the following initial value problem:

$$y'' - 4y' + 4y = \delta(t-4), \quad y(0) = 0, \quad y'(0) = 1$$

Problem 46. Solve the initial value problem:

$$y' - 5y = 1 + u(t-3), \quad y(0) = 0$$

Problem 47. Solve the equation:

$$y(t) - \int_0^t e^{-v}\, y(t-v)\, dv = 3$$

Problem 48. Solve the initial value problem:

$$y'' + y = 4\delta(t-2\pi), \quad y(0) = 1, \quad y'(0) = 0$$

Problem 49. Use the Laplace transform method to solve the initial value problem:

$$y'' - 2y' - 3y = 8e^t, \quad y(0) = 0, \quad y'(0) = 0$$

Problem 50. Solve the initial value problem:

$$y'' + 2y' + y = \delta(t-3), \quad y(0) = 1, \quad y'(0) = -1$$

Problem 51. Find the following information for the function:

$$g(t) = e^{3(t-2)}u(t-2)$$

Determine the values of $g(1)$, $g(3)$, and the Laplace transform $\mathcal{L}\{g(t)\}$.

Problem 52. Solve the initial value problem:

$$y' + 3y = 1 + u(t-2), \quad y(0) = 0$$

Problem 53. Solve the initial value problem:

$$y'' - 2y' + 5y = 2\delta(t-3), \quad y(0) = 0, \quad y'(0) = 2$$

Problem 54. Solve the integro-differential equation

$$y' + 2y - \int_0^t 2\, y(\tau)\sin(t-\tau)\, d\tau = -\text{cos}t$$

subject to the initial condition $y(0) = 0$.

Problem 55. Solve the initial value problem

$$y'' + y = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 1$$

and compute the numerical values of $y\left(\frac{\pi}{2}\right)$ and $y\left(\frac{3\pi}{2}\right)$.

Problem 56. Solve the integral-differential equation using the method of Laplace transforms:

$$y'(t) + 2y(t) - 2\int_0^t y\,(v)\sin(t - v)\,dv = \cos(t), \quad y(0) = 0$$

Problem 57. Solve the initial value problem using the method of Laplace transforms:

$$y''(t) - 4y'(t) + 5y(t) = \delta(t - 3), \quad y(0) = 0, \quad y'(0) = 1$$

Problem 58. Solve the initial value problem using the method of Laplace transforms:

$$y'' + 5y' + 6y = t\delta(t - 3), \quad y(0) = 0, \quad y'(0) = 0$$

Problem 59. Solve the integral equation using the method of Laplace transforms:

$$(\dagger) \quad y(t) + \int_0^t (t - v)y(v)\,dv = 2t$$

Problem 60. Solve the initial value problem using the method of Laplace transforms:

$$y'' + 2y' + 2y = g(t), \quad y(0) = y'(0) = 0$$

where $g(t) = 0$ for $t \leq 2$ and $g(t) = 1$ for $t > 2$.

9.1/5.1 - Introductions to Systems of Linear Differential Equations, Interconnected Fluid Tanks

Problem 61. Two tanks initially hold 200 gallons of pure water each. A mixture with concentration 5 oz/gal flows into Tank 1 at 5 gal/min. The mixture drains from Tank 1 into Tank 2 at 2 gal/min and into the environment at 1 gal/min. Tank 2 drains into the environment at 2 gal/min. Let $Q_1(t)$ and $Q_2(t)$ denote the salt (oz) in Tanks 1 and 2. Set up, but do not solve, a system of differential equations with initial conditions modeling the process.

Problem 62. Consider two interconnected tanks. Tank 1 initially contains 40 grams of sugar dissolved in 60 liters of water, and Tank 2 initially contains 50 liters of water with 25 grams of sugar. Water containing 2 grams of sugar per liter flows into Tank 2 at 3 liters per minute. The mixture drains from Tank 2 at 4 liters per minute, with 1.5 liters per minute flowing into Tank 1 and the rest leaving the system. The mixture in Tank 1 flows back into

Tank 2 at 1.5 liters per minute. If $S_1(t)$ and $S_2(t)$ denote the amounts of sugar in Tanks 1 and 2, respectively, set up, but do not solve, an initial value problem modeling the process.

Problem 63. Two very large tanks initially hold 100 gallons of pure water each. A mixture with concentration 3 oz/gal flows into Tank 1 at 5 gal/min. The mixture drains from Tank 1 into the environment at 2 gal/min and into Tank 2 at 3 gal/min. Another mixture with concentration 4 oz/gal flows into Tank 2 at 2 gal/min. Tank 2 drains into the environment at 2 gal/min. If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in Tanks 1 and 2, respectively, set up, but do not solve, the differential equations and initial conditions modeling the process.

Problem 64. (a) Transform the fourth-order differential equation

$$y^{(4)} - y'' - y = \sin(t^2)$$

into an equivalent system of first-order differential equations.

(b) Express this first-order system in vector-matrix notation.

Do not attempt to solve either the system or the original differential equation.

Problem 65. Consider a system of two interconnected tanks. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 5 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

(a) Set up, but do not solve, an initial value problem modeling the amount of salt in Tank 1 and Tank 2, respectively, at any time t.

(b) For which times t is your model in part (a) valid? Explain why this is so.

Problem 66. Consider two tanks holding initially 100 gallons of pure water each. A mixture of salt and water at a concentration of 4 ounces per gallon flows into Tank 1 at a rate of 5 gallons per minute. The well-stirred mixture in Tank 1 drains into the environment at a rate of 3 gallons per minute, and from Tank 1 into Tank 2 at a rate of 2 gallons per minute. Another mixture of salt and water at a concentration of 5 ounces per gallon flows into Tank 2 at a rate of 2 gallons per minute. The well-stirred mixture in Tank 2 drains into the environment at a rate of 2 gallons per minute.

If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in ounces at time t in Tanks 1 and 2, respectively, SET UP, BUT DO NOT SOLVE, the differential equations and initial conditions that model the flow process.

Problem 67. Consider two interconnected tanks. Tank 1 initially contains 25 gal of water and 20 oz of salt, and Tank 2 initially contains 15 gal of water and 10 oz of salt. Water containing 2 oz/gal of salt flows into Tank 1 at a rate of 2.5 gal/min. The mixture flows from

Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 4 oz/gal of salt also flows into Tank 2 at a rate of 2 gal/min from outside. The mixture drains from Tank 2 at a rate of 5 gal/min, of which some flows back into Tank 1 at a rate of 2.5 gal/min, while the remainder leaves the system.
Set up, BUT DO NOT SOLVE, an initial value problem which models the amount of salt in each tank at all future times.

Problem 68. Consider a system of two interconnected tanks. Tank 1 contains initially 20 gallons of pure water, and Tank 2 initially contains 5 gallons of water and 20 oz of salt. A mixture of salt and water at a concentration of 5 ounces per gallon flows into Tank 1 at a rate of 2 gallons per minute. The well-stirred mixture in Tank 1 drains into Tank 2 at a rate of 3 gallons per minute. A mixture of salt and water at a concentration of 1 ounce per gallon flows into Tank 2 at a rate of 4 gallons per minute, and the well-stirred mixture in Tank 2 drains at a rate of 7 gallons per minute, of which some flows back into Tank 1 at a rate of 2 gallons per minute, while the remainder leaves the system.
If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in ounces at time t in Tanks 1 and 2, respectively, SET UP, BUT DO NOT SOLVE, an initial value problem modeling the amount of salt in Tank 1 and Tank 2 at any time t .

Problem 69. Two interconnected tanks are each initially filled with 100 gallons of pure water. A mixture containing 2 pounds of salt per gallon of water is pumped into Tank 1 at a rate of 3 gal/min, and into Tank 2 at a rate of 2 gal/min. The well-stirred mixture in Tank 1 flows through a pipe into Tank 2 at a rate of 4 gal/min. Also, the well-stirred mixture in Tank 2 flows through another pipe into Tank 1 at a rate of 5 gal/min and into the environment at a rate of 1 gal/min.

1. Draw and label a simple diagram of the system.
2. Find the volume of each tank at any time $t > 0$ (in minutes).
3. With Q_1 and Q_2 denoting the salt content in tanks 1 and 2, in pounds, set up, but DO NOT SOLVE, the differential equations modeling the system.

Problem 70. Consider an interconnected system of two tanks, Tank 1 and Tank 2.

- Tank 1 initially contains 1 gal of water and 2 oz of salt.
- Tank 2 initially contains 1 gal of water and no salt.

The mixture flows from Tank 1 to Tank 2 at a rate of 1 gal/min. The mixture flows from Tank 2 to Tank 1 at a rate of 1 gal/min. Let $x(t)$ and $y(t)$ be the amount of salt (in oz) in Tank 1 and Tank 2, respectively, at time t.

Set up a system of differential equations, including initial conditions, which models the flow process.

9.2-9.3 - Systems and Linear Algebra Problems

Problem 71. If

$$A(t) = \begin{pmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{pmatrix},$$

compute

$$\frac{d}{dt}(A^{-1}(t)).$$

Problem 72. If

$$A(t) = \begin{pmatrix} 2e^{2t} & e^{-t} \\ 2e^{t} & 3e^{-2t} \end{pmatrix},$$

find

$$\frac{d}{dt}(A^{-1}(t)).$$

Problem 73. Verify that the matrix function

$$X(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

satisfies the differential equation

$$X' = AX, \quad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}.$$

Problem 74. Let

$$A(t) = \begin{pmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{pmatrix}.$$

Does

$$\frac{d}{dt}(A^2)$$

equal $2A\frac{dA}{dt}$, or $2\frac{dA}{dt}A$, or $A\frac{dA}{dt} + \frac{dA}{dt}A$, or none of these? Show work supporting your answer.

Problem 75. Are the vectors

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Linearly independent? Show work to justify your answer

Problem 76. a) Transform the following differential equation into a system of first order differential equations:

$$y^{(4)} + 3y^{(3)} - y' + y = t^{-1}sin(t)$$

b) Rewrite the system in matrix/vector form

Problem 77. Given that $x_1(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $x_2(t) = \begin{bmatrix} e^t \\ 3e^{-t} \end{bmatrix}$ are two linearly independent solutions of

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x$$

Find the fundamental matrix, X, such that $X(0) = I$

Problem 78. Let $x_1 = y,\ x_2 = y',\ x_3 = y'',\ x_4 = y'''$. Then solve the following differential equation by transforming it into a system of first-order differential equations:

$$y^{(4)} + 2y'' + 6y = t^3.$$

Problem 79 (11). Find constants α and β so that

$$\mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

is a solution of

$$\mathbf{x}'(t) = \begin{pmatrix} \alpha & -2 \\ 2 & \beta \end{pmatrix} \mathbf{x}(t).$$

Problem 80 (11). Determine whether the following vectors are linearly dependent or independent. If they are linearly dependent, find a linear relation among them.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 81.

1. Transform the following differential equation

   $$y^{(4)} - e^t y''' + 2y = t\cos(t)$$

   into a system of first order differential equations.

2. Write the system that you obtain in part (a) in matrix-vector form.

Problem 82. Complete the problems below.

1. Express the differential equations

   $$x'' + 3x' - y' + 2y = 0, \qquad y'' + x' + 3y' + y = 0$$

as a first order system in normal form. Then write the system in the form

$$\mathbf{x}' = A\mathbf{x}.$$

2. Determine whether the following vector functions are linearly dependent or linearly independent. Justify your answer.

$$\mathbf{x}_1 = \begin{pmatrix} e^t \\ 0 \\ 2e^t \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} -e^t \\ e^t \\ 0 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} e^t \\ e^t \\ 4e^t \end{pmatrix}.$$

Problem 83. Rewrite the scalar higher order linear differential equation

$$y''' + 2y'' + 4y = 11t$$

in matrix-vector form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Problem 84. Express the system of differential equations

$$\begin{aligned} x' &= 2x + 3y + z \\ y' &= 4z - 5y \\ z' &= 3y - 2x \end{aligned}$$

in the form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and determine the values for the entries of matrix A.

Problem 85. Express the following system of differential equations

$$\begin{cases} \dot{x}_1 = 3x_1 + 2x_2 \\ \dot{x}_2 = 4x_1 + 5x_2 \end{cases}$$

in the matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and determine the values of $a, b, c,$ and d.

Problem 86. Let $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find a matrix A such that the system of differential equations

$$\begin{cases} x' = y + 2x + z \\ y' = 2z - x \\ z' = 4x \end{cases}$$

is equivalent to $\mathbf{u}' = A\mathbf{u}$.

Problem 87. Express the differential equation

$$y'''(t) + 3y'(t) - 20y(t) = 0$$

as a system of first order differential equations. (Do NOT write the system in matrix form.)

Problem 88. Express the system of differential equations

$$\begin{aligned} x' &= 4x + y + 2z \\ y' &= -3z - 7y \\ z' &= 5y - 6x \end{aligned}$$

in matrix form by finding the matrix A so that:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Problem 89. Express the system of differential equations

$$\begin{aligned} x_1' &= x_1 + 2x_3 \\ x_2' &= 9x_3 - 6x_2 + 3x_1 \\ x_3' &= -4x_2 + 5x_3 \end{aligned}$$

in the form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and determine the values for the entries of matrix A.

Problem 90. Express the system of differential equations

$$\begin{aligned} x' &= x + y + z \\ y' &= 2z - x \\ z' &= 4y \end{aligned}$$

in matrix form by finding the matrix A so that:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Problem 91. Express the damped mass-spring oscillator equation

$$(*) \quad my'' + by' + ky = 0$$

as a system of first order differential equations. Do NOT write the system in matrix form.

9.4 - 9.5 - Homogeneous Systems with Constant Coefficients

Problem 92. Find the general solution of the system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}.$$

Problem 93. Find the solution of the initial value problem

$$\begin{aligned} x_1' &= x_1 - x_2, \quad x_1(0) = 1, \\ x_2' &= x_1 + x_2, \quad x_2(0) = 2. \end{aligned}$$

Problem 94. Find the solution of the system

$$\begin{aligned} \frac{dx}{dt} &= x + 6y, \\ \frac{dy}{dt} &= x - 4y, \end{aligned} \quad x(0) = 5, \quad y(0) = 9.$$

Problem 95. Solve the initial value problem

$$\begin{aligned} x' &= x + y, \\ y' &= 4x + y, \end{aligned} \quad x(0) = 2, \quad y(0) = 0.$$

Problem 96. Find the general solution of the system

$$\begin{aligned} x' &= 3x + 6y, \\ y' &= -x - 2y. \end{aligned}$$

Problem 97. Find the general solution of

$$x' = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} x$$

Problem 98. Solve the initial value problem

$$\begin{aligned} x' &= -5x + y, \quad x(0) = 2, \\ y' &= -3x - y, \quad y(0) = -1. \end{aligned}$$

Describe the behavior of the solution as $t \to \infty$.

Problem 99. (a) Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) Describe the behavior of the solution as $t \to \infty$.

Problem 100. Find two linearly independent vector solutions of

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix},$$

and calculate their Wronskian.

Problem 101. Given that $\Phi(t) = \begin{bmatrix} e^{-t} & e^{t} \\ 3e^{-t} & e^{t} \end{bmatrix}$ is a fundamental matrix for

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

Solve the initial value problem

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Problem 102. Solve the initial value problem

$$\begin{aligned} x'(t) &= -2x - y \\ y'(t) &= -7x + 4y \end{aligned}$$

with $x(0) = 1, y(0) = 3$

Problem 103. Find the general solution of the system of differential equations

$$x' = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} x$$

Problem 104. Find the general solution of the system of differential equations

$$x' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} x$$

Problem 105. Find the general solution of the system

$$\frac{dx_1}{dt} = -2x_2 + x_1, \qquad \frac{dx_2}{dt} = 3x_1 - 4x_2.$$

Problem 106. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Problem 107. Find the general solution of the following system of differential equations:

$$\begin{cases} x_1'(t) = x_1(t) + 2x_2(t), \\ x_2'(t) = 2x_1(t) + x_2(t). \end{cases}$$

Problem 108. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & 2 \\ 10 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Problem 109. Find the general solution to the differential equation system:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -1 & 1 \\ 8 & 1 \end{pmatrix}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Problem 110. Find the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Problem 111. Find the general solution of the differential equation system:

$$\vec{x}' = A\vec{x}, \quad A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Problem 112. Find the general solution of the differential equation system:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

Problem 113. Find the general solution of the differential equation system:

$$\vec{x}' = A\vec{x}, \quad A = \begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}$$

Problem 114. Find the general solution of the differential equation system:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Problem 115. Given that $\mathbf{x} = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}$ is a solution of the following system

$$\mathbf{x}' = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} \mathbf{x},$$

find the general solution of system (1).

Problem 116. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$$

Problem 117. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

Exam 3 Review Sheet

Math 3304

7.5-7.9 - Solving Initial Value Problems with Laplace Transforms

Problem 1. Find the solution of the initial value problem

$$y'' + 2y' + 2y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 0.$$

Then compute the value of the solution, accurate to five decimal places, when $t = 6$.

Solution 1. Taking the Laplace transform of both sides, and using Row 13 and Row 20 from the table, we get

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\delta(t-5)\}.$$

Since $y(0) = 0$ and $y'(0) = 0$,

$$s^2Y(s) + 2sY(s) + 2Y(s) = e^{-5s}.$$

Thus

$$(s^2 + 2s + 2)Y(s) = e^{-5s},$$

so

$$Y(s) = \frac{e^{-5s}}{s^2 + 2s + 2}.$$

Completing the square gives

$$s^2 + 2s + 2 = (s+1)^2 + 1,$$

so

$$Y(s) = e^{-5s}\frac{1}{(s+1)^2 + 1}.$$

Using Row 9 from the table,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 1}\right\} = e^{-t}\sin(t).$$

Then by Row 16,

$$y(t) = u(t-5)e^{-(t-5)}\sin(t-5).$$

Therefore,

$$y(6) = e^{-1}\sin(1).$$

Hence

$$y(6) \approx 0.30956.$$

Problem 2. Solve the initial value problem

$$y'' - 4y' + 5y = \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Express your final answer without any unit step functions.

Solution 2. Taking the Laplace transform of both sides, and using Row 13 and Row 20 from the table, we get

$$s^2Y(s) - 4sY(s) + 5Y(s) = e^{-2\pi s}.$$

Thus

$$(s^2 - 4s + 5)Y(s) = e^{-2\pi s},$$

so

$$Y(s) = \frac{e^{-2\pi s}}{s^2 - 4s + 5}.$$

Completing the square gives

$$s^2 - 4s + 5 = (s-2)^2 + 1,$$

so

$$Y(s) = e^{-2\pi s}\frac{1}{(s-2)^2 + 1}.$$

Using Row 9 from the table,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2 + 1}\right\} = e^{2t}\sin(t).$$

Using Row 16 with $a = 2\pi$, we get

$$y(t) = e^{2(t-2\pi)}\sin(t - 2\pi), \quad t \geq 2\pi.$$

Since the forcing does not occur until $t = 2\pi$, we have

$$y(t) = 0, \quad t < 2\pi.$$

Therefore,

$$y(t) = \begin{cases} 0, & t < 2\pi, \\ e^{2(t-2\pi)}\sin(t-2\pi), & t \geq 2\pi. \end{cases}$$

Problem 3. Solve the integrodifferential equation

$$y'(t) = 1 - \int_0^t e^{-2\tau}\, y(t-\tau)d\tau, \quad y(0) = 1.$$

Solution 3. Taking the Laplace transform of both sides, and using Row 12 and Row 19 from the table, we get

$$sY(s) - 1 = \frac{1}{s} - \mathcal{L}\{e^{-2t}\}Y(s).$$

Using Row 2 from the table,

$$\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}.$$

Thus

$$sY(s) - 1 = \frac{1}{s} - \frac{1}{s+2}Y(s).$$

Solving for $Y(s)$, we get

$$\left(s + \frac{1}{s+2}\right)Y(s) = 1 + \frac{1}{s}.$$

Therefore,

$$Y(s) = \frac{s+1}{s} \cdot \frac{s+2}{s(s+2)+1}.$$

Since

$$s(s+2) + 1 = s^2 + 2s + 1 = (s+1)^2,$$

we have

$$Y(s) = \frac{(s+1)(s+2)}{s(s+1)^2} = \frac{s+2}{s(s+1)}.$$

Using partial fractions,

$$\frac{s+2}{s(s+1)} = \frac{2}{s} - \frac{1}{s+1}.$$

Using Rows 1 and 2 from the table, we get

$$y(t) = 2 - e^{-t}.$$

Problem 4. Find the solution of the integro-differential equation

$$y'(t) + 3\int_0^t e^{-4r}\, y(t-r)\, dr = 1, \quad y(0) = 0.$$

Solution 4. Taking the Laplace transform of both sides, and using Row 12 and Row 19 from the table, we get

$$sY(s) + 3\mathcal{L}\{e^{-4t}\}Y(s) = \frac{1}{s}.$$

Using Row 2 from the table,

$$\mathcal{L}\{e^{-4t}\} = \frac{1}{s+4}.$$

Thus

$$sY(s) + \frac{3}{s+4}Y(s) = \frac{1}{s}.$$

Solving for $Y(s)$, we get

$$\left(s + \frac{3}{s+4}\right)Y(s) = \frac{1}{s}.$$

Therefore,

$$Y(s) = \frac{s+4}{s(s^2+4s+3)}.$$

Factoring gives

$$Y(s) = \frac{s+4}{s(s+1)(s+3)}.$$

Using partial fractions,

$$\frac{s+4}{s(s+1)(s+3)} = \frac{4}{3s} - \frac{3}{2(s+1)} + \frac{1}{6(s+3)}.$$

Using Rows 1 and 2 from the table, we get

$$y(t) = \frac{4}{3} - \frac{3}{2}e^{-t} + \frac{1}{6}e^{-3t}.$$

Problem 5. Solve the initial value problem

$$y'' + 4y = 1 - u(t-3\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Solution 5. Taking the Laplace transform of both sides, and using Row 13 and Row 17 from the table, we get

$$(s^2+4)Y(s)=\frac{1}{s}-\frac{e^{-3\pi s}}{s}.$$

Thus

$$Y(s)=\frac{1}{s(s^2+4)}-\frac{e^{-3\pi s}}{s(s^2+4)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+4)}=\frac{1}{4s}-\frac{s}{4(s^2+4)}.$$

Using Rows 1 and 5 from the table,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}=\frac{1}{4}-\frac{1}{4}\cos(2t).$$

Using Row 16, we get

$$y(t)=\frac{1}{4}(1-\cos(2t))-\frac{1}{4}u(t-3\pi)(1-\cos(2(t-3\pi))).$$

Since

$$\cos(2(t-3\pi))=\cos(2t-6\pi)=\cos(2t),$$

we can write the solution as

$$y(t)=\begin{cases}\frac{1}{4}(1-\cos(2t)), & t<3\pi,\\ 0, & t\geq 3\pi.\end{cases}$$

Problem 6. Solve the initial value problem

$$y''-2y'+2y=\delta(t-2),\quad y(0)=1,\quad y'(0)=-2.$$

Solution 6. Taking the Laplace transform of both sides, and using Row 13 and Row 20 from the table, we get

$$s^2Y(s)-s+2-2(sY(s)-1)+2Y(s)=e^{-2s}.$$

Thus

$$(s^2-2s+2)Y(s)-s+4=e^{-2s},$$

so

$$Y(s)=\frac{s-4}{s^2-2s+2}+\frac{e^{-2s}}{s^2-2s+2}.$$

Completing the square gives

$$s^2 - 2s + 2 = (s-1)^2 + 1.$$

Therefore,

$$Y(s) = \frac{s-1}{(s-1)^2+1} - \frac{3}{(s-1)^2+1} + e^{-2s}\frac{1}{(s-1)^2+1}.$$

Using Rows 5 and 9 from the table,

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} = e^t\cos(t),$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} = e^t\sin(t).$$

Using Row 16, we get

$$y(t) = e^t\cos(t) - 3e^t\sin(t) + u(t-2)e^{t-2}\sin(t-2).$$

Problem 7. Solve the integral equation

$$y(t) + \int_0^t e^{-\tau}\, y(t-\tau)\, d\tau = 1.$$

Solution 7. Taking the Laplace transform of both sides, and using Row 19 from the table, we get

$$Y(s) + \mathcal{L}\{e^{-t}\}Y(s) = \frac{1}{s}.$$

Using Row 2 from the table,

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}.$$

Thus

$$Y(s) + \frac{1}{s+1}Y(s) = \frac{1}{s}.$$

Solving for $Y(s)$, we get

$$\left(1 + \frac{1}{s+1}\right)Y(s) = \frac{1}{s}.$$

Therefore,

$$Y(s) = \frac{s+1}{s(s+2)}.$$

Using partial fractions,

$$\frac{s+1}{s(s+2)} = \frac{1}{2s} + \frac{1}{2(s+2)}.$$

Using Rows 1 and 2 from the table, we get

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}.$$

Problem 8. (a) Find the solution $y = y(t)$ of the initial value problem

$$y'' + 3y' + 2y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 1.$$

(b) Which is greater, $y(6)$ or $y(1)$? Justify your answer.

Solution 8. Taking the Laplace transform of both sides, and using Row 13 and Row 20 from the table, we get

$$s^2Y(s) - 1 + 3sY(s) + 2Y(s) = e^{-5s}.$$

Thus

$$(s^2 + 3s + 2)Y(s) = 1 + e^{-5s},$$

so

$$Y(s) = \frac{1}{(s+1)(s+2)} + \frac{e^{-5s}}{(s+1)(s+2)}.$$

Using partial fractions,

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Using Row 2 from the table,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = e^{-t} - e^{-2t}.$$

Using Row 16, we get

$$y(t) = e^{-t} - e^{-2t} + u(t-5)\left(e^{-(t-5)} - e^{-2(t-5)}\right).$$

Thus

$$y(1) = e^{-1} - e^{-2}.$$

Also,

$$y(6) = e^{-6} - e^{-12} + e^{-1} - e^{-2}.$$

Since

$$e^{-6} - e^{-12} > 0,$$

we have

$$y(6) > y(1).$$

Problem 9. Solve the integral equation

$$y(t) = 3t^2 - e^{-t} - \int_0^t y\,(u)e^{t-u}\,du.$$

Solution 9. Taking the Laplace transform of both sides, and using Row 19 from the table, we get

$$Y(s) = \frac{6}{s^3} - \frac{1}{s+1} - Y(s)\mathcal{L}\{e^t\}.$$

Using Row 2 from the table,

$$\mathcal{L}\{e^t\} = \frac{1}{s-1}.$$

Thus

$$Y(s) = \frac{6}{s^3} - \frac{1}{s+1} - \frac{1}{s-1}Y(s).$$

Solving for $Y(s)$, we get

$$\left(1 + \frac{1}{s-1}\right)Y(s) = \frac{6}{s^3} - \frac{1}{s+1}.$$

Therefore,

$$Y(s) = \frac{s-1}{s}\left(\frac{6}{s^3} - \frac{1}{s+1}\right).$$

So

$$Y(s) = \frac{6}{s^3} - \frac{6}{s^4} - \frac{s-1}{s(s+1)}.$$

Using partial fractions,

$$-\frac{s-1}{s(s+1)} = \frac{1}{s} - \frac{2}{s+1}.$$

Thus

$$Y(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}.$$

Using Rows 1, 2, and 4 from the table, we get

$$y(t) = 3t^2 - t^3 + 1 - 2e^{-t}.$$

Problem 10. Find the solution of the initial value problem

$$y'' + 4y = 2 - 4u(t - \frac{\pi}{2}), \quad y(0) = 0, \quad y'(0) = 0.$$

Solution 10. Taking the Laplace transform of both sides, and using Row 13 and Row 17 from the table, we get

$$(s^2 + 4)Y(s) = \frac{2}{s} - \frac{4e^{-\frac{\pi}{2}s}}{s}.$$

Thus

$$Y(s) = \frac{2}{s(s^2 + 4)} - \frac{4e^{-\frac{\pi}{2}s}}{s(s^2 + 4)}.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 4)} = \frac{1}{4s} - \frac{s}{4(s^2 + 4)}.$$

Using Rows 1 and 5 from the table,

$$\mathcal{L}^{-1}\left\{\frac{2}{s(s^2 + 4)}\right\} = \frac{1}{2}(1 - \cos(2t)).$$

Also,

$$\mathcal{L}^{-1}\left\{\frac{4}{s(s^2 + 4)}\right\} = 1 - \cos(2t).$$

Using Row 16, we get

$$y(t) = \frac{1}{2}(1 - \cos(2t)) - u\left(t - \frac{\pi}{2}\right)\left(1 - \cos\left(2\left(t - \frac{\pi}{2}\right)\right)\right).$$

Since

$$\cos\left(2\left(t - \frac{\pi}{2}\right)\right) = \cos(2t - \pi) = -\cos(2t),$$

we get

$$y(t) = \frac{1}{2}(1 - \cos(2t)) - u\left(t - \frac{\pi}{2}\right)(1 + \cos(2t)).$$

Problem 11. Find the solution of the initial value problem

$$y'' + y = \frac{1}{\pi}\delta(t-\pi), \quad y(0) = 0, \quad y'(0) = -1.$$

Write your solution as a piecewise defined function and sketch its graph on the interval

$$0 \le t \le 3\pi.$$

Solution 11. Taking the Laplace transform of both sides, and using Row 13 and Row 20 from the table, we get

$$s^2 Y(s) + 1 + Y(s) = \frac{1}{\pi}e^{-\pi s}.$$

Thus

$$(s^2+1)Y(s) = \frac{1}{\pi}e^{-\pi s} - 1,$$

so

$$Y(s) = -\frac{1}{s^2+1} + \frac{1}{\pi}e^{-\pi s}\frac{1}{s^2+1}.$$

Using Row 9 from the table,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t).$$

Using Row 16, we get

$$y(t) = -\sin(t) + \frac{1}{\pi}u(t-\pi)\sin(t-\pi).$$

Since

$$\sin(t-\pi) = -\sin(t),$$

we can write the solution as

$$y(t) = \begin{cases} -\sin(t), & 0 \le t < \pi, \\ -\left(1+\frac{1}{\pi}\right)\sin(t), & t \ge \pi. \end{cases}$$

Problem 12. Solve the integral equation

$$y(t) + 5\int_0^t y\,(t-r)\cos(2r)\,dr = e^{-3t}.$$

Solution 12. Taking the Laplace transform of both sides, and using Row 19 from the table, we get

$$Y(s) + 5Y(s)\mathcal{L}\{\cos(2t)\} = \frac{1}{s+3}.$$

Using Rows 2 and 5 from the table,

$$\mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4}.$$

Thus

$$Y(s) + \frac{5s}{s^2+4}Y(s) = \frac{1}{s+3}.$$

Solving for $Y(s)$, we get

$$\left(1 + \frac{5s}{s^2+4}\right)Y(s) = \frac{1}{s+3}.$$

Therefore,

$$Y(s) = \frac{s^2+4}{(s+3)(s^2+5s+4)}.$$

Factoring gives

$$Y(s) = \frac{s^2+4}{(s+3)(s+1)(s+4)}.$$

Using partial fractions,

$$\frac{s^2+4}{(s+3)(s+1)(s+4)} = \frac{5}{6(s+1)} + \frac{13}{2(s+3)} - \frac{19}{3(s+4)}.$$

Using Row 2 from the table, we get

$$y(t) = \frac{5}{6}e^{-t} + \frac{13}{2}e^{-3t} - \frac{19}{3}e^{-4t}.$$

Problem 13. Compute the inverse Laplace transform of

$$F(s) = \frac{2s^2+6s+12}{s(s-2)(s+2)}.$$

Solution 13. Using partial fractions, we write

$$\frac{2s^2+6s+12}{s(s-2)(s+2)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2}.$$

Thus

$$2s^2+6s+12 = A(s-2)(s+2) + Bs(s+2) + Cs(s-2).$$

Using $s = 0$, we get

$$12 = -4A,$$

so

$$A = -3.$$

Using $s = 2$, we get

$$32 = 8B,$$

so

$$B = 4.$$

Using $s = -2$, we get

$$8 = 8C,$$

so

$$C = 1.$$

Therefore,

$$F(s) = -\frac{3}{s} + \frac{4}{s-2} + \frac{1}{s+2}.$$

Using Rows 1 and 2 from the table, we get

$$\mathcal{L}^{-1}\{F(s)\} = -3 + 4e^{2t} + e^{-2t}.$$

Problem 14. (a) Solve the initial value problem

$$y'' + 4y = g(t), \quad y(0) = 1, \quad y'(0) = 0,$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 3\pi, \\ 1, & 3\pi \le t < \infty. \end{cases}$$

(b) Which is greater, $y(2\pi)$ or $y(5\pi)$? Justify your answer.

Solution 14. We can write

$$g(t) = u(t - 3\pi).$$

Taking the Laplace transform of both sides, and using Row 13 and Row 17 from the table, we get

$$s^2Y(s) - s + 4Y(s) = \frac{e^{-3\pi s}}{s}.$$

Thus

$$(s^2+4)Y(s) = s + \frac{e^{-3\pi s}}{s},$$

so

$$Y(s) = \frac{s}{s^2+4} + e^{-3\pi s}\frac{1}{s(s^2+4)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+4)} = \frac{1}{4s} - \frac{s}{4(s^2+4)}.$$

Using Rows 1 and 5 from the table,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{4}(1-\cos(2t)).$$

Using Row 16, we get

$$y(t) = \cos(2t) + \frac{1}{4}u(t-3\pi)(1-\cos(2(t-3\pi))).$$

Since

$$\cos(2(t-3\pi)) = \cos(2t-6\pi) = \cos(2t),$$

we have

$$y(t) = \cos(2t) + \frac{1}{4}u(t-3\pi)(1-\cos(2t)).$$

Thus

$$y(2\pi) = \cos(4\pi) = 1.$$

Also,

$$y(5\pi) = \cos(10\pi) + \frac{1}{4}(1-\cos(10\pi)) = 1.$$

Therefore,

$$y(2\pi) = y(5\pi).$$

Problem 15. Solve the initial value problem

$$y^{(4)} - y = 2\delta(t-1), \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

Solution 15. Taking the Laplace transform of both sides, and using Row 15 and Row 20 from the table, we get

$$s^4Y(s) - Y(s) = 2e^{-s}.$$

Thus

$$(s^4 - 1)Y(s) = 2e^{-s},$$

so

$$Y(s) = \frac{2e^{-s}}{s^4 - 1}.$$

Factoring gives

$$Y(s) = 2e^{-s}\frac{1}{(s-1)(s+1)(s^2+1)}.$$

Using partial fractions,

$$\frac{2}{(s-1)(s+1)(s^2+1)} = \frac{1}{2(s-1)} - \frac{1}{2(s+1)} - \frac{1}{s^2+1}.$$

Using Rows 2 and 9 from the table,

$$\mathcal{L}^{-1}\left\{\frac{2}{s^4-1}\right\} = \frac{1}{2}e^t - \frac{1}{2}e^{-t} - \sin(t).$$

Using Row 16, we get

$$y(t) = u(t-1)\left(\frac{1}{2}e^{t-1} - \frac{1}{2}e^{-(t-1)} - \sin(t-1)\right).$$

Problem 16. (a) Find the solution of the initial value problem

$$y'' + y = g(t), \quad y(0) = 1, \quad y'(0) = -1,$$

where

$$g(t) = \begin{cases} 1-t, & t < 1, \\ 0, & t \geq 1. \end{cases}$$

(b) Draw the graph of the solution on the interval

$$0 \leq t \leq 3\pi.$$

Solution 16. We can write

$$g(t) = 1 - t + u(t-1)(t-1).$$

Taking the Laplace transform of both sides, and using Rows 4, 13, and 16 from the table, we get

$$s^2Y(s) - s + 1 + Y(s) = \frac{1}{s} - \frac{1}{s^2} + e^{-s}\frac{1}{s^2}.$$

Thus

$$(s^2+1)Y(s) = s - 1 + \frac{1}{s} - \frac{1}{s^2} + e^{-s}\frac{1}{s^2},$$

so

$$Y(s) = \frac{s-1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{1}{s^2(s^2+1)} + e^{-s}\frac{1}{s^2(s^2+1)}.$$

Simplifying the non-shifted terms gives

$$\frac{s-1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{1}{s^2(s^2+1)} = \frac{1}{s} - \frac{1}{s^2}.$$

Also,

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

Using Rows 4, 9, and 16 from the table, we get

$$y(t) = 1 - t + u(t-1)((t-1) - \sin(t-1)).$$

Therefore,

$$y(t) = \begin{cases} 1 - t, & 0 \le t < 1, \\ -\sin(t-1), & t \ge 1. \end{cases}$$

Problem 17. Solve the integro-differential equation

$$y'(t) + 2y(t) - 2\int_0^t y(\xi)\sin(t-\xi)\,d\xi = -\sin(t),$$

subject to the initial condition

$$y(0) = 1.$$

Solution 17. Taking the Laplace transform of both sides, and using Rows 9, 12, and 19 from the table, we get

$$sY(s) - 1 + 2Y(s) - 2Y(s)\frac{1}{s^2+1} = -\frac{1}{s^2+1}.$$

Thus

$$\left(s + 2 - \frac{2}{s^2+1}\right)Y(s) = 1 - \frac{1}{s^2+1}.$$

Therefore,

$$Y(s) = \frac{\frac{s^2}{s^2+1}}{\frac{(s+2)(s^2+1)-2}{s^2+1}}.$$

So

$$Y(s) = \frac{s^2}{(s+2)(s^2+1)-2}.$$

Expanding and factoring gives

$$Y(s) = \frac{s^2}{s^3+2s^2+s} = \frac{s}{s^2+2s+1}.$$

Thus

$$Y(s) = \frac{s}{(s+1)^2} = \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2}.$$

Using Rows 2 and 8 from the table, we get

$$y(t) = e^{-t} - te^{-t}.$$

Therefore,

$$y(t) = (1-t)e^{-t}.$$

Problem 18. (a) Solve the initial value problem

$$y'' + 4y = \delta(t-\pi), \quad y(0) = 1/2, \quad y'(0) = 0.$$

(b) Which is greater, $y(\pi)$ or $y(3\pi/2)$? Justify your answer.

Solution 18. Taking the Laplace transform of both sides, and using Rows 13 and 20 from the table, we get

$$s^2Y(s) - \frac{s}{2} + 4Y(s) = e^{-\pi s}.$$

Thus

$$(s^2+4)Y(s) = \frac{s}{2} + e^{-\pi s},$$

so

$$Y(s) = \frac{s}{2(s^2+4)} + e^{-\pi s}\frac{1}{s^2+4}.$$

Using Rows 5, 9, and 16 from the table, we get

$$y(t) = \frac{1}{2}\cos(2t) + \frac{1}{2}u(t-\pi)\sin(2(t-\pi)).$$

Since

$$\sin(2(t-\pi)) = \sin(2t - 2\pi) = \sin(2t),$$

we have

$$y(t) = \frac{1}{2}\cos(2t) + \frac{1}{2}u(t-\pi)\sin(2t).$$

Thus

$$y(\pi) = \frac{1}{2}\cos(2\pi) + \frac{1}{2}\sin(2\pi) = \frac{1}{2}.$$

Also,

$$y\left(\frac{3\pi}{2}\right) = \frac{1}{2}\cos(3\pi) + \frac{1}{2}\sin(3\pi) = -\frac{1}{2}.$$

Therefore,

$$y(\pi) > y\left(\frac{3\pi}{2}\right).$$

Problem 19. Solve the initial value problem

$$y'' + 4y' + 13y = h(t), \quad y(0) = y'(0) = 0,$$

where

$$h(t) = \begin{cases} 0, & 0 \le t < \pi, \\ 13, & \pi \le t < \infty. \end{cases}$$

Solution 19. We can write

$$h(t) = 13u(t-\pi).$$

Taking the Laplace transform of both sides, and using Rows 13 and 17 from the table, we get

$$(s^2 + 4s + 13)Y(s) = \frac{13e^{-\pi s}}{s}.$$

Thus

$$Y(s) = e^{-\pi s}\frac{13}{s(s^2 + 4s + 13)}.$$

Completing the square gives

$$s^2 + 4s + 13 = (s+2)^2 + 9.$$

Using partial fractions,

$$\frac{13}{s((s+2)^2+9)} = \frac{1}{s} - \frac{s+4}{(s+2)^2+9}.$$

So

$$\frac{13}{s((s+2)^2+9)} = \frac{1}{s} - \frac{s+2}{(s+2)^2+9} - \frac{2}{(s+2)^2+9}.$$

Using Rows 1, 5, and 9 from the table,

$$\mathcal{L}^{-1}\left\{\frac{13}{s((s+2)^2+9)}\right\} = 1 - e^{-2t}\cos(3t) - \frac{2}{3}e^{-2t}\sin(3t).$$

Using Row 16, we get

$$y(t) = u(t-\pi)\left(1 - e^{-2(t-\pi)}\cos(3(t-\pi)) - \frac{2}{3}e^{-2(t-\pi)}\sin(3(t-\pi))\right).$$

Problem 20. Find the solution $y(t)$ of the integral equation

$$y(t) - 9\int_0^t (t-\tau)y(\tau)\,d\tau = 9t^2.$$

Solution 20. Taking the Laplace transform of both sides, and using Rows 4 and 19 from the table, we get

$$Y(s) - 9\mathcal{L}\{t\}Y(s) = \frac{18}{s^3}.$$

Using Row 4 from the table,

$$\mathcal{L}\{t\} = \frac{1}{s^2}.$$

Thus

$$Y(s) - \frac{9}{s^2}Y(s) = \frac{18}{s^3}.$$

Solving for $Y(s)$, we get

$$\left(1 - \frac{9}{s^2}\right)Y(s) = \frac{18}{s^3}.$$

Therefore,

$$Y(s) = \frac{18}{s(s^2-9)}.$$

Using partial fractions,

$$\frac{18}{s(s-3)(s+3)} = -\frac{2}{s} + \frac{1}{s-3} + \frac{1}{s+3}.$$

Using Rows 1 and 2 from the table, we get

$$y(t) = -2 + e^{3t} + e^{-3t}.$$

Problem 21. Solve the initial value problem

$$y'' + 3y' + 2y = u(t-10), \quad y(0) = y'(0) = 0$$

Solution 21. Take Laplace Transforms of both sides: (Use entries 13, 12, 17)

$$(s^2Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) + 2Y(s) = \frac{e^{-10s}}{s}$$

Use the initial conditions $y(0) = y'(0) = 0$

$$\begin{aligned} s^2Y(s) + 3sY(s) + 2Y(s) &= \frac{e^{-10s}}{s} \\ Y(s)(s^2 + 3s + 2) &= \frac{e^{-10s}}{s} \\ Y(s)(s+1)(s+2) &= \frac{e^{-10s}}{s} \\ Y(s) &= e^{-10s}\frac{1}{s(s+1)(s+2)} \end{aligned}$$

Use partial fraction decomposition on right fraction. We obtain that:

$$\frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

so,

$$Y(s) = e^{-10s}\left(\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)$$

Let

$$F(s) = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

Taking inverse laplace of $F(s)$ we obtain:

$$f(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

Then, using entry 16

$$\begin{aligned}
\mathcal{L}^{-1}(Y(s)) &= \mathcal{L}^{-1}\left(e^{-10s}\left(\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)\right) \\
&= \mathcal{L}^{-1}(e^{-10s}F(s)) \\
&= u(t-10)f(t-10) \\
&= u(t-10)\left(\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right)
\end{aligned}$$

Problem 22. Find the solution of the integral equation

$$y(t) - \int_0^t y\,(\tau)sin(t-\tau)d\tau = \delta(t-\pi)cos(t)$$

Solution 22. Take the Laplace Transform of both sides. For the Integral, we have $f(t) = y(t), g(t-\tau) = sin(t-\tau) \Rightarrow g(t) = sin(t)$. Then,

$$\begin{aligned}
Y(s) - \left(\frac{1}{s^2+1}\right)Y(s) &= cos(\pi)e^{-\pi s} \\
\left(1 - \frac{1}{s^2+1}\right)Y(s) &= -e^{-\pi s} \\
\left(\frac{s^2}{s^2+1}\right)Y(s) &= -e^{-\pi s} \\
Y(s) &= -\left(\frac{s^2+1}{s^2}\right)e^{-\pi s} \\
Y(s) &= -\left(1 + \frac{1}{s^2}\right)e^{-\pi s} \\
Y(s) &= -e^{-\pi s} - \frac{1}{s^2}e^{-\pi s}
\end{aligned}$$

By taking the inverse Laplace transform, using entry 20 for the first term and entry 16 for the second term we have:

$$y(t) = -\delta(t-\pi) - u(t-\pi)(t-\pi)$$

Problem 23. Solve the initial value problem

$$y' + 3\int_0^t e^{-4(t-\tau)}\,y(\tau)d\tau = 0, \qquad y(0) = 1$$

Solution 23. Take the Laplace Transform of both sides. For the integral, we have $f(t) = e^{-4t}$ and $g(t) = y(t)$. Then,

$$\begin{aligned}
\mathcal{L}\{y'\} + 3\mathcal{L}\left\{\int_0^t e^{-4(t-\tau)}\, y(\tau)\, d\tau\right\} &= 0 \\
sY(s) - y(0) + 3\left(\frac{1}{s+4}\right)Y(s) &= 0 \\
sY(s) - 1 + \frac{3}{s+4}Y(s) &= 0 \\
\left(s + \frac{3}{s+4}\right)Y(s) &= 1 \\
Y(s) &= \frac{1}{s + \frac{3}{s+4}} \\
Y(s) &= \frac{s+4}{s(s+4)+3} \\
Y(s) &= \frac{s+4}{s^2+4s+3} \\
Y(s) &= \frac{s+4}{(s+1)(s+3)}.
\end{aligned}$$

Now use partial fractions:

$$\begin{aligned}
\frac{s+4}{(s+1)(s+3)} &= \frac{A}{s+1} + \frac{B}{s+3} \\
s + 4 &= A(s+3) + B(s+1).
\end{aligned}$$

Setting $s = -1$ gives

$$3 = 2A \quad \Rightarrow \quad A = \frac{3}{2}.$$

Setting $s = -3$ gives

$$1 = -2B \quad \Rightarrow \quad B = -\frac{1}{2}.$$

Therefore,

$$Y(s) = \frac{3}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}.$$

Taking the inverse Laplace transform gives

$$y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}.$$

Problem 24. Solve the initial value problem

$$y''(t) + 2y'(t) + 5y(t) = 2\delta(t - \frac{\pi}{2})sin(t), \qquad y(0) = 1, y'(0) = -1$$

Solution 24. Take the Laplace transform of both sides of the differential equation. Using the linearity property and the transforms for derivatives from **Entry #12** and **Entry #13**:

$$\mathcal{L}\{y''(t)\} + 2\mathcal{L}\{y'(t)\} + 5\mathcal{L}\{y(t)\} = \mathcal{L}\{2\delta(t - \pi/2)\sin(t)\}$$

Apply the initial conditions $y(0) = 1$ and $y'(0) = -1$:

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) = 2\mathcal{L}\{\delta(t - \pi/2)\sin(t)\}$$

$$[s^2Y(s) - s(1) - (-1)] + 2[sY(s) - 1] + 5Y(s) = 2\mathcal{L}\{\delta(t - \pi/2)\sin(t)\}$$

$$[s^2Y(s) - s + 1] + 2sY(s) - 2 + 5Y(s) = 2\mathcal{L}\{\delta(t - \pi/2)\sin(t)\}$$

For the right side, use **Entry #21**: $\mathcal{L}\{f(t)\delta(t - a)\} = f(a)e^{-as}$. Here $f(t) = \sin(t)$ and $a = \pi/2$:

$$2\mathcal{L}\{\sin(t)\delta(t - \pi/2)\} = 2\sin(\pi/2)e^{-\pi s/2} = 2(1)e^{-\pi s/2} = 2e^{-\pi s/2}$$

Substitute this back and group the $Y(s)$ terms:

$$Y(s)(s^2 + 2s + 5) - s - 1 = 2e^{-\pi s/2}$$

Solve for $Y(s)$:

$$Y(s)(s^2 + 2s + 5) = s + 1 + 2e^{-\pi s/2} \Rightarrow Y(s) = \frac{s + 1}{s^2 + 2s + 5} + \frac{2e^{-\pi s/2}}{s^2 + 2s + 5}$$

Complete the square for the denominator: $s^2 + 2s + 5 = (s + 1)^2 + 4 = (s + 1)^2 + 2^2$.

$$Y(s) = \frac{s + 1}{(s + 1)^2 + 2^2} + e^{-\pi s/2} \cdot \frac{2}{(s + 1)^2 + 2^2}$$

Now, take the inverse Laplace transform:

- For $\frac{s+1}{(s+1)^2+2^2}$: Using **Entry #10** with $a = -1$ and $b = 2$, the inverse is $e^{-t}\cos(2t)$.
- For $e^{-\pi s/2} \cdot \frac{2}{(s+1)^2+2^2}$: Let $H(s) = \frac{2}{(s+1)^2+2^2}$. From **Entry #9** ($a = -1, b = 2$), its inverse is $h(t) = e^{-t}\sin(2t)$. Using **Entry #16** for the shift $e^{-as}H(s)$, the inverse is $h(t - a)u(t - a)$ with $a = \pi/2$:

  $$e^{-(t-\pi/2)}\sin(2(t - \pi/2))u(t - \pi/2) = e^{-(t-\pi/2)}\sin(2t - \pi)u(t - \pi/2)$$

  Since $\sin(2t - \pi) = -\sin(2t)$, this term simplifies to $-e^{-(t-\pi/2)}\sin(2t)u(t - \pi/2)$.

The final solution is:

$$y(t) = e^{-t}\cos(2t) - e^{-(t-\pi/2)}\sin(2t)u(t - \pi/2)$$

Problem 25. Solve the initial value problem

$$y'' + 9y = f(t), y'(0) = 1$$

where

$$f(t) = \begin{cases} 3 & 0 \leq t < 2 \\ 0 & 2 \leq t < \infty \end{cases}$$

Solution 25. First rewrite $f(t)$ using the unit step function:

$$f(t) = 3 - 3u(t-2).$$

Take Laplace Transforms of both sides:

$$(s^2Y(s) - sy(0) - y'(0)) + 9Y(s) = \frac{3}{s} - \frac{3e^{-2s}}{s}.$$

Use the initial conditions $y(0) = 0$ and $y'(0) = 1$.

$$\begin{aligned} s^2Y(s) - 1 + 9Y(s) &= \frac{3}{s} - \frac{3e^{-2s}}{s} \\ (s^2+9)Y(s) &= 1 + \frac{3}{s} - \frac{3e^{-2s}}{s} \\ Y(s) &= \frac{1}{s^2+9} + \frac{3}{s(s^2+9)} - e^{-2s}\frac{3}{s(s^2+9)}. \end{aligned}$$

Use partial fraction decomposition:

$$\frac{3}{s(s^2+9)} = \frac{\frac{1}{3}}{s} - \frac{\frac{1}{3}s}{s^2+9}.$$

Therefore,

$$Y(s) = \frac{1}{s^2+9} + \left(\frac{\frac{1}{3}}{s} - \frac{\frac{1}{3}s}{s^2+9}\right) - e^{-2s}\left(\frac{\frac{1}{3}}{s} - \frac{\frac{1}{3}s}{s^2+9}\right).$$

Let

$$F(s) = \frac{\frac{1}{3}}{s} - \frac{\frac{1}{3}s}{s^2+9}.$$

Taking inverse Laplace of $F(s)$, we obtain

$$f_1(t) = \frac{1}{3} - \frac{1}{3}\cos(3t).$$

Also,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin(3t).$$

Then, using the shifting formula,

$$\begin{aligned} y(t) &= \frac{1}{3}\sin(3t) + f_1(t) - u(t-2)f_1(t-2) \\ &= \frac{1}{3}\sin(3t) + \frac{1}{3} - \frac{1}{3}\cos(3t) - u(t-2)\left(\frac{1}{3} - \frac{1}{3}\cos(3(t-2))\right). \end{aligned}$$

Problem 26. Solve the Integro-differential Equation

$$y' + \int_0^t c\,os(t-\tau)y(\tau)d\tau = 1, \quad y(0) = 0$$

Solution 26. Take the Laplace Transform of both sides. For the integral, we have $f(t) = \cos(t)$ and $g(t) = y(t)$. Then,

$$\begin{aligned} \mathcal{L}\{y'\} + \mathcal{L}\left\{\int_0^t \cos(t-\tau)y(\tau)\,d\tau\right\} &= \mathcal{L}\{1\} \\ sY(s) - y(0) + \left(\frac{s}{s^2+1}\right)Y(s) &= \frac{1}{s} \\ sY(s) + \frac{s}{s^2+1}Y(s) &= \frac{1}{s} \\ \left(s + \frac{s}{s^2+1}\right)Y(s) &= \frac{1}{s} \\ \left(\frac{s(s^2+2)}{s^2+1}\right)Y(s) &= \frac{1}{s} \\ Y(s) &= \frac{s^2+1}{s^2(s^2+2)}. \end{aligned}$$

Now use partial fractions:

$$\begin{aligned} \frac{s^2+1}{s^2(s^2+2)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+2} \\ s^2+1 &= As(s^2+2) + B(s^2+2) + (Cs+D)s^2. \end{aligned}$$

Comparing coefficients gives

$$A = 0, \qquad B = \frac{1}{2}, \qquad C = 0, \qquad D = \frac{1}{2}.$$

Therefore,

$$Y(s) = \frac{1}{2}\frac{1}{s^2} + \frac{1}{2}\frac{1}{s^2+2}.$$

Taking the inverse Laplace transform gives

$$y(t) = \frac{1}{2}t + \frac{1}{2\sqrt{2}}\sin(\sqrt{2}t).$$

Problem 27. Given that $\mathcal{L}(t sin(at)) = \frac{2as}{(s^2+a^2)^2}$, solve the IVP

$$y'' + 9y = \delta(t - 3\pi) + cos(3t), \quad y(0) = y'(0) = 0$$

Calculate the value of $y(\frac{2\pi}{3})$.

Solution 27. Take Laplace Transforms of both sides:

$$(s^2Y(s) - sy(0) - y'(0)) + 9Y(s) = e^{-3\pi s} + \frac{s}{s^2 + 9}.$$

Use the initial conditions $y(0) = y'(0) = 0$.

$$\begin{aligned} s^2Y(s) + 9Y(s) &= e^{-3\pi s} + \frac{s}{s^2 + 9} \\ (s^2 + 9)Y(s) &= e^{-3\pi s} + \frac{s}{s^2 + 9} \\ Y(s) &= \frac{e^{-3\pi s}}{s^2 + 9} + \frac{s}{(s^2 + 9)^2}. \end{aligned}$$

Now split the inverse Laplace transform into two parts. For the first term,

$$\frac{1}{s^2 + 9} = \frac{1}{3}\frac{3}{s^2 + 9}.$$

Let

$$F(s) = \frac{1}{s^2 + 9}.$$

Then

$$f(t) = \frac{1}{3}\sin(3t).$$

Using the shifting formula,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s^2 + 9}\right\} &= u(t - 3\pi)f(t - 3\pi) \\ &= \frac{1}{3}u(t - 3\pi)\sin(3(t - 3\pi)). \end{aligned}$$

For the second term, use the given formula

$$\mathcal{L}\{t\sin(at)\} = \frac{2as}{(s^2 + a^2)^2}.$$

With $a = 3$, we have

$$\mathcal{L}\{t\sin(3t)\} = \frac{6s}{(s^2+9)^2}.$$

Therefore,

$$\frac{s}{(s^2+9)^2} = \frac{1}{6}\frac{6s}{(s^2+9)^2}.$$

Taking the inverse Laplace transform gives

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = \frac{1}{6}t\sin(3t).$$

Thus,

$$y(t) = \frac{1}{3}u(t-3\pi)\sin(3(t-3\pi)) + \frac{1}{6}t\sin(3t).$$

Now calculate $y\left(\frac{2\pi}{3}\right)$. Since

$$\frac{2\pi}{3} < 3\pi,$$

we have

$$u\left(\frac{2\pi}{3} - 3\pi\right) = 0.$$

Therefore,

$$\begin{aligned} y\left(\frac{2\pi}{3}\right) &= \frac{1}{6}\left(\frac{2\pi}{3}\right)\sin\left(3\cdot\frac{2\pi}{3}\right) \\ &= \frac{\pi}{9}\sin(2\pi) \\ &= 0. \end{aligned}$$

Problem 28. Find the solution of the integral equation

$$y(t) - \int_0^t \tau\, y(t-\tau)d\tau = e^t$$

Solution 28. Take the Laplace Transform of both sides. For the integral, we have $f(t) = t$ and $g(t) = y(t)$. Then,

$$\mathcal{L}\{y(t)\} - \mathcal{L}\left\{\int_0^t \tau\, y(t-\tau)\, d\tau\right\} = \mathcal{L}\{e^t\}$$

$$Y(s) - \left(\frac{1}{s^2}\right)Y(s) = \frac{1}{s-1}$$

$$\left(1 - \frac{1}{s^2}\right)Y(s) = \frac{1}{s-1}$$

$$\left(\frac{s^2-1}{s^2}\right)Y(s) = \frac{1}{s-1}$$

$$Y(s) = \frac{s^2}{(s-1)(s^2-1)}$$

$$Y(s) = \frac{s^2}{(s-1)^2(s+1)}.$$

Now use partial fractions:

$$\frac{s^2}{(s-1)^2(s+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1}$$

$$s^2 = A(s-1)(s+1) + B(s+1) + C(s-1)^2.$$

Setting $s = 1$ gives

$$1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}.$$

Setting $s = -1$ gives

$$1 = 4C \quad \Rightarrow \quad C = \frac{1}{4}.$$

Using $s = 0$ gives

$$0 = -A + \frac{1}{2} + \frac{1}{4} \quad \Rightarrow \quad A = \frac{3}{4}.$$

Therefore,

$$Y(s) = \frac{3}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2} + \frac{1}{4}\frac{1}{s+1}.$$

Taking the inverse Laplace transform gives

$$y(t) = \frac{3}{4}e^t + \frac{1}{2}te^t + \frac{1}{4}e^{-t}.$$

Problem 29. Solve the initial value problem

$$y'' + y = u_\pi(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

Solution 29. Take Laplace Transforms of both sides:

$$\left(s^2Y(s) - sy(0) - y'(0)\right) + Y(s) = \frac{e^{-\pi s}}{s}.$$

Use the initial conditions $y(0) = 0$ and $y'(0) = 1$.

$$\begin{aligned} s^2Y(s) - 1 + Y(s) &= \frac{e^{-\pi s}}{s} \\ (s^2+1)Y(s) &= 1 + \frac{e^{-\pi s}}{s} \\ Y(s) &= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s(s^2+1)}. \end{aligned}$$

Use partial fraction decomposition:

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Therefore,

$$Y(s) = \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right).$$

Let

$$F(s) = \frac{1}{s} - \frac{s}{s^2+1}.$$

Taking inverse Laplace of $F(s)$, we obtain

$$f(t) = 1 - \cos(t).$$

Then, using the shifting formula,

$$\begin{aligned} y(t) &= \sin(t) + u(t-\pi)f(t-\pi) \\ &= \sin(t) + u(t-\pi)(1 - \cos(t-\pi)). \end{aligned}$$

Problem 30. Solve the integro-differential equation

$$2y'(t) = \int_0^t (t-\tau)^2 y(\tau)\, d\tau - 2t$$

with

$$y(0) = 1.$$

Solution 30. Take Laplace Transforms of both sides. For the integral, we have $f(t) = t^2$ and $g(t) = y(t)$.

Therefore,

$$2(sY(s) - y(0)) = \frac{2}{s^3}Y(s) - \frac{2}{s^2}.$$

Use the initial condition $y(0) = 1$.

$$\begin{aligned}
2(sY(s) - 1) &= \frac{2}{s^3}Y(s) - \frac{2}{s^2} \\
2sY(s) - 2 &= \frac{2}{s^3}Y(s) - \frac{2}{s^2} \\
sY(s) - 1 &= \frac{1}{s^3}Y(s) - \frac{1}{s^2} \\
sY(s) - \frac{1}{s^3}Y(s) &= 1 - \frac{1}{s^2} \\
\left(s - \frac{1}{s^3}\right)Y(s) &= \frac{s^2 - 1}{s^2} \\
\left(\frac{s^4 - 1}{s^3}\right)Y(s) &= \frac{s^2 - 1}{s^2} \\
Y(s) &= \frac{s(s^2 - 1)}{s^4 - 1}.
\end{aligned}$$

Since

$$s^4 - 1 = (s^2 - 1)(s^2 + 1),$$

we get

$$\begin{aligned}
Y(s) &= \frac{s(s^2 - 1)}{(s^2 - 1)(s^2 + 1)} \\
&= \frac{s}{s^2 + 1}.
\end{aligned}$$

Taking the inverse Laplace transform gives

$$y(t) = \cos(t).$$

Problem 31. Solve the initial value problem

$$y'' + 3y' + 2y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t. \end{cases}$$

Solution 31. First rewrite $g(t)$ using the unit step function:

$$g(t) = 1 - u(t - 10).$$

Take Laplace Transforms of both sides:

$$(s^2Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}.$$

Use the initial conditions $y(0) = 0$ and $y'(0) = 0$.

$$\begin{aligned}
s^2Y(s) + 3sY(s) + 2Y(s) &= \frac{1}{s} - \frac{e^{-10s}}{s} \\
(s^2 + 3s + 2)Y(s) &= \frac{1}{s} - \frac{e^{-10s}}{s} \\
(s+1)(s+2)Y(s) &= \frac{1}{s} - \frac{e^{-10s}}{s} \\
Y(s) &= \frac{1}{s(s+1)(s+2)} - e^{-10s}\frac{1}{s(s+1)(s+2)}.
\end{aligned}$$

Use partial fraction decomposition:

$$\frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}.$$

Let

$$F(s) = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}.$$

Taking inverse Laplace of $F(s)$, we obtain

$$f(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

Therefore,

$$Y(s) = F(s) - e^{-10s}F(s).$$

Then, using the shifting formula (entry 16),

$$\begin{aligned}
y(t) &= f(t) - u(t-10)f(t-10) \\
&= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u(t-10)\left(\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right).
\end{aligned}$$

Problem 32. Solve the integral equation

$$y(t) + 4\int_0^t (t-\tau)y(\tau)\,d\tau = t\,u_1(t) - t.$$

Solution 32. Take the Laplace Transform of both sides. For the integral, we have $f(t) = t$ and $g(t) = y(t)$. Also,

$$tu(t-1) - t = (t-1)u(t-1) + u(t-1) - t.$$

Then,

$$\begin{aligned}
\mathcal{L}\{y(t)\} + 4\mathcal{L}\left\{\int_0^t (t-\tau)y(\tau)\,d\tau\right\} &= \mathcal{L}\{tu_1(t) - t\} \\
Y(s) + 4\left(\frac{1}{s^2}\right)Y(s) &= e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - \frac{1}{s^2} \\
\left(1 + \frac{4}{s^2}\right)Y(s) &= e^{-s}\left(\frac{s+1}{s^2}\right) - \frac{1}{s^2} \\
\left(\frac{s^2+4}{s^2}\right)Y(s) &= e^{-s}\left(\frac{s+1}{s^2}\right) - \frac{1}{s^2} \\
Y(s) &= e^{-s}\frac{s+1}{s^2+4} - \frac{1}{s^2+4}.
\end{aligned}$$

Rewrite $Y(s)$ as

$$Y(s) = e^{-s}\left(\frac{s}{s^2+4} + \frac{1}{s^2+4}\right) - \frac{1}{s^2+4}.$$

Taking the inverse Laplace transform gives

$$y(t) = u_1(t)\left[\cos(2(t-1)) + \frac{1}{2}\sin(2(t-1))\right] - \frac{1}{2}\sin(2t).$$

Problem 33. Solve the initial value problem

$$y'' - 4y' + 5y = \delta(t-\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution 33. Take Laplace Transforms of both sides:

$$(s^2Y(s) - sy(0) - y'(0)) - 4(sY(s) - y(0)) + 5Y(s) = e^{-\pi s}.$$

Use the initial conditions $y(0) = 0$ and $y'(0) = 0$.

$$\begin{aligned}
s^2Y(s) - 4sY(s) + 5Y(s) &= e^{-\pi s} \\
(s^2 - 4s + 5)Y(s) &= e^{-\pi s} \\
((s-2)^2 + 1)Y(s) &= e^{-\pi s} \\
Y(s) &= e^{-\pi s}\frac{1}{(s-2)^2+1}.
\end{aligned}$$

Let

$$F(s) = \frac{1}{(s-2)^2+1}.$$

Taking inverse Laplace of $F(s)$, we obtain

$$f(t) = e^{2t}\sin(t).$$

Then, using the shifting formula,

$$\begin{aligned} y(t) &= u(t-\pi)f(t-\pi) \\ &= u(t-\pi)e^{2(t-\pi)}\sin(t-\pi). \end{aligned}$$

Problem 34. Find the solution of the integral equation

$$y(t) - \int_0^t \sin(t-\tau)y(\tau)\,d\tau = t.$$

Solution 34. Take the Laplace Transform of both sides. For the integral, we have $f(t) = \sin(t)$ and $g(t) = y(t)$. Then,

$$\begin{aligned} \mathcal{L}\{y(t)\} - \mathcal{L}\left\{\int_0^t \sin(t-\tau)y(\tau)\,d\tau\right\} &= \mathcal{L}\{t\} \\ Y(s) - \left(\frac{1}{s^2+1}\right)Y(s) &= \frac{1}{s^2} \\ \left(1 - \frac{1}{s^2+1}\right)Y(s) &= \frac{1}{s^2} \\ \left(\frac{s^2}{s^2+1}\right)Y(s) &= \frac{1}{s^2} \\ Y(s) &= \frac{s^2+1}{s^4} \\ Y(s) &= \frac{1}{s^2} + \frac{1}{s^4}. \end{aligned}$$

Taking the inverse Laplace transform gives

$$y(t) = t + \frac{t^3}{6}.$$

Problem 35. Use the Laplace transform to solve the initial value problem

$$y' - y = f(t), \qquad y(0) = 1,$$

where

$$f(t) = \begin{cases} 0, & 0 \le t < 4, \\ e^{3(t-4)}, & t \ge 4. \end{cases}$$

Solution 35. First rewrite $f(t)$ using the unit step function:

$$f(t) = u(t-4)e^{3(t-4)}.$$

Take Laplace Transforms of both sides:

$$(sY(s) - y(0)) - Y(s) = e^{-4s}\frac{1}{s-3}.$$

Use the initial condition $y(0) = 1$.

$$\begin{aligned} sY(s) - 1 - Y(s) &= e^{-4s}\frac{1}{s-3} \\ (s-1)Y(s) - 1 &= e^{-4s}\frac{1}{s-3} \\ (s-1)Y(s) &= 1 + e^{-4s}\frac{1}{s-3} \\ Y(s) &= \frac{1}{s-1} + e^{-4s}\frac{1}{(s-1)(s-3)}. \end{aligned}$$

Use partial fraction decomposition:

$$\frac{1}{(s-1)(s-3)} = -\frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s-3}.$$

Let

$$F(s) = -\frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s-3}.$$

Taking inverse Laplace of $F(s)$, we obtain

$$f_1(t) = -\frac{1}{2}e^t + \frac{1}{2}e^{3t}.$$

Then, using the shifting formula (entry 16),

$$\begin{aligned} y(t) &= e^t + u(t-4)f_1(t-4) \\ &= e^t + u(t-4)\left(-\frac{1}{2}e^{t-4} + \frac{1}{2}e^{3(t-4)}\right). \end{aligned}$$

Problem 36. Solve the integral equation

$$y(t) + 3\int_0^t y(\tau)\sin(t-\tau)\,d\tau = t.$$

Solution 36. Take the Laplace Transform of both sides. For the integral, we have $f(t) = y(t)$ and $g(t) = \sin(t)$. Then,

$$\mathcal{L}\{y(t)\} + 3\mathcal{L}\left\{\int_0^t y(\tau)\sin(t-\tau)\,d\tau\right\} = \mathcal{L}\{t\}$$
$$Y(s) + 3Y(s)\left(\frac{1}{s^2+1}\right) = \frac{1}{s^2}$$
$$\left(1 + \frac{3}{s^2+1}\right)Y(s) = \frac{1}{s^2}$$
$$\left(\frac{s^2+4}{s^2+1}\right)Y(s) = \frac{1}{s^2}$$
$$Y(s) = \frac{s^2+1}{s^2(s^2+4)}.$$

Now use partial fractions:

$$\frac{s^2+1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$
$$s^2 + 1 = As(s^2+4) + B(s^2+4) + (Cs+D)s^2.$$

Comparing coefficients gives

$$A = 0, \qquad B = \frac{1}{4}, \qquad C = 0, \qquad D = \frac{3}{4}.$$

Therefore,

$$Y(s) = \frac{1}{4}\frac{1}{s^2} + \frac{3}{4}\frac{1}{s^2+4}.$$

Taking the inverse Laplace transform gives

$$y(t) = \frac{1}{4}t + \frac{3}{8}\sin(2t).$$

Problem 37. Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t-\pi), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution 37. Take Laplace Transforms of both sides:

$$(s^2Y(s) - sy(0) - y'(0)) + 2(sY(s) - y(0)) + 2Y(s) = e^{-\pi s}.$$

Use the initial conditions $y(0) = 1$ and $y'(0) = 1$.

$$(s^2Y(s) - s - 1) + 2(sY(s) - 1) + 2Y(s) = e^{-\pi s}$$
$$(s^2 + 2s + 2)Y(s) - s - 3 = e^{-\pi s}$$
$$(s^2 + 2s + 2)Y(s) = s + 3 + e^{-\pi s}$$
$$Y(s) = \frac{s+3}{s^2+2s+2} + e^{-\pi s}\frac{1}{s^2+2s+2}.$$

Complete the square:

$$s^2 + 2s + 2 = (s+1)^2 + 1.$$

Rewrite $Y(s)$:

$$Y(s) = \frac{s+1}{(s+1)^2+1} + \frac{2}{(s+1)^2+1} + e^{-\pi s}\frac{1}{(s+1)^2+1}.$$

Taking inverse Laplace transforms of the first two terms gives

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} = e^{-t}\cos(t),$$

and

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+1}\right\} = 2e^{-t}\sin(t).$$

Let

$$F(s) = \frac{1}{(s+1)^2+1}.$$

Then

$$f(t) = e^{-t}\sin(t).$$

Therefore, using the shifting formula,

$$\begin{aligned} y(t) &= e^{-t}\cos(t) + 2e^{-t}\sin(t) + u(t-\pi)f(t-\pi) \\ &= e^{-t}\cos(t) + 2e^{-t}\sin(t) + u(t-\pi)e^{-(t-\pi)}\sin(t-\pi). \end{aligned}$$

Problem 38. Find $g(t)$, which is the following inverse Laplace transform:

$$g(t) = \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2+4}\right\}$$

Solution 38. To find the inverse transform, we first identify the component without the exponential shift. Let:

$$F(s) = \frac{1}{s^2+4}$$

We recognize this as being related to the sine transform from **Entry #4** of the table, where $\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}$. Here, $b^2 = 4$, so $b = 2$. We adjust the constant in the numerator:

$$F(s) = \frac{1}{2}\cdot\frac{2}{s^2+2^2}$$

Taking the inverse Laplace transform of $F(s)$ gives:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\sin(2t)$$

Next, we account for the e^{-3s} term using **Entry #16** from the table, which states that $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$. In this case, $a = 3$.

Applying the time-shifting property:

$$g(t) = f(t-3)u(t-3)$$

Substituting our expression for $f(t)$ into the shifted form:

$$g(t) = \frac{1}{2}\sin(2(t-3))u(t-3)$$

The final solution is:

$$g(t) = \frac{1}{2}\sin(2t-6)u(t-3)$$

Problem 39. Make use of Laplace transform to solve the following initial value problem:

$$y'' + 3y' + 2y = 1, \quad y(0) = 0, \quad y'(0) = 0$$

Solution 39. First, we take the Laplace transform of both sides of the differential equation. Using **Entry #12** and **Entry #13** for the derivatives and **Entry #1** for the constant 1, we have:

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{1\}$$

Substituting the formulas for the transforms of derivatives:

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s}$$

Applying the initial conditions $y(0) = 0$ and $y'(0) = 0$:

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s}$$

Factoring out $Y(s)$:

$$(s^2 + 3s + 2)Y(s) = \frac{1}{s} \Longrightarrow Y(s) = \frac{1}{s(s^2 + 3s + 2)}$$

Factoring the quadratic term $s^2 + 3s + 2 = (s+1)(s+2)$, we get:

$$Y(s) = \frac{1}{s(s+1)(s+2)}$$

We use Partial Fraction Decomposition (PFD) to split this into simpler terms:

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Multiplying by the denominator gives $1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$.

- Let $s = 0$: $1 = A(1)(2) \Rightarrow A = \frac{1}{2}$
- Let $s = -1$: $1 = B(-1)(1) \Rightarrow B = -1$
- Let $s = -2$: $1 = C(-2)(-1) \Rightarrow C = \frac{1}{2}$

Now we have:

$$Y(s) = \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{s+1} + \frac{1}{2} \cdot \frac{1}{s+2}$$

Taking the inverse Laplace transform using **Entry #1** and **Entry #2**:

$$y(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

The final solution is:

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

Problem 40. Solve the following integral equation:

$$y(t) - \int_0^t e^{-(t-v)}\, y(v)\, dv = u(t-1)$$

Solution 40. The integral in the equation is a convolution. Let $h(t) = e^{-t}$. Then the integral is $(h * y)(t) = \int_0^t h\,(t-v)y(v)\,dv$. We can rewrite the equation as:

$$y(t) - (e^{-t} * y(t)) = u(t-1)$$

Taking the Laplace transform of both sides and applying the **Convolution Theorem (Entry #19)**:

$$\mathcal{L}\{y(t)\} - \mathcal{L}\{e^{-t}\} \cdot \mathcal{L}\{y(t)\} = \mathcal{L}\{u(t-1)\}$$

Using **Entry #2** for the exponential and **Entry #17** for the unit step function:

$$Y(s) - \left(\frac{1}{s+1}\right)Y(s) = \frac{e^{-s}}{s}$$

Factor out $Y(s)$:

$$Y(s)\left[1 - \frac{1}{s+1}\right] = \frac{e^{-s}}{s}$$

Simplify the bracketed term:

$$Y(s)\left[\frac{(s+1)-1}{s+1}\right] = Y(s)\left[\frac{s}{s+1}\right] = \frac{e^{-s}}{s}$$

Solving for $Y(s)$:

$$Y(s) = \frac{e^{-s}}{s} \cdot \frac{s+1}{s} = e^{-s}\left(\frac{s+1}{s^2}\right) = e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$$

Let $F(s) = \frac{1}{s} + \frac{1}{s^2}$. Taking the inverse transform using **Entry #1** and **Entry #3**:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^2}\right\} = 1 + t$$

Finally, we account for the e^{-s} shift using the property from **Entry #16**, $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$, where $a = 1$:

$$y(t) = f(t-1)u(t-1) = [1 + (t-1)]u(t-1)$$

The final solution is:

$$y(t) = tu(t-1)$$

Problem 41. Solve the given initial value problem using Laplace transforms:

$$y'' - y' - 2y = 0, \quad y(0) = -2, \quad y'(0) = 5$$

Solution 41. First, we take the Laplace transform of both sides of the differential equation using **Entry #12** and **Entry #13** from the table:

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

Substituting the derivative formulas:

$$[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 2Y(s) = 0$$

Applying the initial conditions $y(0) = -2$ and $y'(0) = 5$:

$$[s^2Y(s) - s(-2) - 5] - [sY(s) - (-2)] - 2Y(s) = 0$$

Simplify the expression:

$$s^2Y(s) + 2s - 5 - sY(s) - 2 - 2Y(s) = 0$$

Group the $Y(s)$ terms:

$$(s^2 - s - 2)Y(s) + 2s - 7 = 0$$

Solve for $Y(s)$:

$$(s^2 - s - 2)Y(s) = 7 - 2s \Rightarrow Y(s) = \frac{7 - 2s}{s^2 - s - 2}$$

Factoring the denominator $s^2 - s - 2 = (s-2)(s+1)$, we use Partial Fraction Decomposition:

$$\frac{7-2s}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

Multiply by the common denominator:

$$7 - 2s = A(s+1) + B(s-2)$$

- Let $s = -1$: $7 - 2(-1) = B(-1-2) \Longrightarrow 9 = -3B \Longrightarrow B = -3$
- Let $s = 2$: $7 - 2(2) = A(2+1) \Longrightarrow 3 = 3A \Longrightarrow A = 1$

Now we have:

$$Y(s) = \frac{1}{s-2} - \frac{3}{s+1}$$

Taking the inverse Laplace transform using **Entry #2** ($e^{at} \leftrightarrow \frac{1}{s-a}$):

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

The final solution is:

$$y(t) = e^{2t} - 3e^{-t}$$

Problem 42. Find the solution of the integral equation:

$$y(t) - \int_0^t \sin(t-\tau) y(\tau)\, d\tau = t$$

Solution 42. The integral term $\int_0^t \sin(t-\tau)y(\tau)\, d\tau$ is the convolution of $\sin(t)$ and $y(t)$. Taking the Laplace transform of both sides and using the **Convolution Theorem (Entry #19)**:

$$\mathcal{L}\{y(t)\} - \mathcal{L}\{\sin(t)\} \cdot \mathcal{L}\{y(t)\} = \mathcal{L}\{t\}$$

Using **Entry #4** for $\sin(t)$ (with $b = 1$) and **Entry #3** for t (with $n = 1$):

$$Y(s) - \frac{1}{s^2+1} Y(s) = \frac{1}{s^2}$$

Factor out $Y(s)$ on the left side:

$$Y(s)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s^2}$$

Combine the terms inside the brackets:

$$Y(s)\left[\frac{(s^2+1)-1}{s^2+1}\right]=\frac{1}{s^2}\Longrightarrow Y(s)\left[\frac{s^2}{s^2+1}\right]=\frac{1}{s^2}$$

Solve for $Y(s)$ by multiplying both sides by $\frac{s^2+1}{s^2}$:

$$Y(s)=\frac{1}{s^2}\cdot\frac{s^2+1}{s^2}=\frac{s^2+1}{s^4}$$

Split the fraction:

$$Y(s)=\frac{s^2}{s^4}+\frac{1}{s^4}=\frac{1}{s^2}+\frac{1}{s^4}$$

Now, take the inverse Laplace transform using **Entry #3**, which states $\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}$:

- For $\frac{1}{s^2}$: $n+1=2\Longrightarrow n=1$. Since $1!=1$, the inverse is $t^1=t$.
- For $\frac{1}{s^4}$: $n+1=4\Longrightarrow n=3$. Since $3!=6$, we rewrite the term as $\frac{1}{6}\cdot\frac{6}{s^4}$. The inverse is $\frac{1}{6}t^3$.

The final solution is:

$$y(t)=t+\frac{1}{6}t^3$$

Problem 43. Solve the initial value problem:

$$2y''+4y'+10y=\delta(t-5),\quad y(0)=0,\quad y'(0)=0$$

Solution 43. First, we take the Laplace transform of both sides of the differential equation. Using **Entry #12** and **Entry #13** for the derivatives and **Entry #20** for the Dirac delta function $\delta(t-a)\leftrightarrow e^{-as}$:

$$2[s^2Y(s)-sy(0)-y'(0)]+4[sY(s)-y(0)]+10Y(s)=e^{-5s}$$

Applying the initial conditions $y(0)=0$ and $y'(0)=0$:

$$(2s^2+4s+10)Y(s)=e^{-5s}$$

Solving for $Y(s)$:

$$Y(s)=e^{-5s}\left(\frac{1}{2s^2+4s+10}\right)=\frac{1}{2}e^{-5s}\left(\frac{1}{s^2+2s+5}\right)$$

To find the inverse transform, we complete the square in the denominator:

$$s^2+2s+5=(s^2+2s+1)+4=(s+1)^2+2^2$$

So we have:

$$Y(s) = \frac{1}{2}e^{-5s}\left(\frac{1}{(s+1)^2 + 2^2}\right)$$

We adjust the numerator to match **Entry #9**, $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$, where $a = -1$ and $b = 2$:

$$Y(s) = \frac{1}{2} \cdot \frac{1}{2}e^{-5s}\left(\frac{2}{(s+1)^2 + 2^2}\right) = \frac{1}{4}e^{-5s}\left(\frac{2}{(s+1)^2 + 2^2}\right)$$

Let $F(s) = \frac{2}{(s+1)^2+2^2}$. From the table, $f(t) = e^{-t}\sin(2t)$.

Finally, we apply the time-shifting property from **Entry #16**, $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$, with $a = 5$:

$$y(t) = \frac{1}{4}f(t-5)u(t-5)$$

Substituting $f(t-5)$:

$$y(t) = \frac{1}{4}e^{-(t-5)}\sin(2(t-5))u(t-5)$$

Problem 44. Use Laplace Transform method to solve the initial value problem:

$$y'' + 4y = 2; \quad y(0) = 0, \quad y'(0) = 0$$

Solution 44. First, we take the Laplace transform of both sides of the equation. Using **Entry #13** for the second derivative and **Entry #1** for the constant 2:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{2\}$$

Substituting the formulas and applying the initial conditions $y(0) = 0$ and $y'(0) = 0$:

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \frac{2}{s}$$

$$(s^2 + 4)Y(s) = \frac{2}{s}$$

Solving for $Y(s)$:

$$Y(s) = \frac{2}{s(s^2+4)}$$

We use Partial Fraction Decomposition (PFD) to split the expression. Since $s^2 + 4$ is an irreducible quadratic, we use the form:

$$\frac{2}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}$$

Multiplying by the common denominator $s(s^2 + 4)$:

$$2 = A(s^2 + 4) + (Bs + C)s$$

$$2 = (A + B)s^2 + Cs + 4A$$

By matching the coefficients on both sides:

- **Constants:** $4A = 2 \Longrightarrow A = \frac{1}{2}$
- s **coefficients:** $C = 0$
- s^2 **coefficients:** $A + B = 0 \Longrightarrow B = -A = -\frac{1}{2}$

Substituting these values back into our $Y(s)$ expression:

$$Y(s) = \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s}{s^2 + 4}$$

Now, we take the inverse Laplace transform. Using **Entry #1** for $\frac{1}{s}$ and **Entry #5** for $\frac{s}{s^2+2^2}$:

$$y(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\}$$

The final solution is:

$$y(t) = \frac{1}{2} - \frac{1}{2}\cos(2t)$$

Problem 45. Solve the following initial value problem:

$$y'' - 4y' + 4y = \delta(t - 4), \quad y(0) = 0, \quad y'(0) = 1$$

Solution 45. We begin by taking the Laplace transform of both sides. Using **Entry #12**, **Entry #13**, and **Entry #20** from the table:

$$[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + 4Y(s) = e^{-4s}$$

Applying the initial conditions $y(0) = 0$ and $y'(0) = 1$:

$$[s^2Y(s) - 0 - 1] - 4[sY(s) - 0] + 4Y(s) = e^{-4s}$$

$$(s^2 - 4s + 4)Y(s) - 1 = e^{-4s}$$

We recognize the quadratic as a perfect square: $s^2 - 4s + 4 = (s - 2)^2$. Substituting this in and solving for $Y(s)$:

$$(s - 2)^2Y(s) = 1 + e^{-4s} \Longrightarrow Y(s) = \frac{1}{(s - 2)^2} + e^{-4s}\frac{1}{(s - 2)^2}$$

To find the inverse transform, let $F(s) = \frac{1}{(s-2)^2}$. According to **Entry #8** (with $n = 1, a = 2$), we have:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} = te^{2t}$$

Now we apply the inverse transform to the full expression. For the second term, we use the time-shifting property from **Entry #16**, $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$, with $a = 4$:

$$y(t) = f(t) + f(t-4)u(t-4)$$

Substituting our expression for $f(t)$ into the result:

$$y(t) = te^{2t} + (t-4)e^{2(t-4)}u(t-4)$$

The final solution is:

$$y(t) = te^{2t} + (t-4)e^{2t-8}u(t-4)$$

Problem 46. Solve the initial value problem:

$$y' - 5y = 1 + u(t-3), \quad y(0) = 0$$

Solution 46. First, we take the Laplace transform of both sides of the differential equation. Using **Entry #12** for the derivative, **Entry #1** for the constant 1, and **Entry #17** for the unit step function $u(t-3)$:

$$\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{1\} + \mathcal{L}\{u(t-3)\}$$

Substituting the formulas and applying the initial condition $y(0) = 0$:

$$[sY(s) - y(0)] - 5Y(s) = \frac{1}{s} + \frac{e^{-3s}}{s}$$

$$(s-5)Y(s) = \frac{1}{s} + \frac{e^{-3s}}{s}$$

Solving for $Y(s)$:

$$Y(s) = \frac{1}{s(s-5)} + e^{-3s}\left(\frac{1}{s(s-5)}\right)$$

Next, we use Partial Fraction Decomposition (PFD) for the term $\frac{1}{s(s-5)}$:

$$\frac{1}{s(s-5)} = \frac{A}{s} + \frac{B}{s-5} \Longrightarrow 1 = A(s-5) + Bs$$

- Let $s = 0$: $1 = A(-5) \Longrightarrow A = -1/5$
- Let $s = 5$: $1 = B(5) \Longrightarrow B = 1/5$

So, $\frac{1}{s(s-5)} = -\frac{1}{5} \cdot \frac{1}{s} + \frac{1}{5} \cdot \frac{1}{s-5}$. Let $F(s) = \frac{1}{s(s-5)}$. Taking the inverse transform of $F(s)$ using **Entry #1** and **Entry #2**:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{5} + \frac{1}{5}e^{5t}$$

Now we apply the inverse transform to the full expression for $Y(s)$. For the second term, we use the time-shifting property from **Entry #16**, $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$, with $a = 3$:

$$y(t) = f(t) + f(t-3)u(t-3)$$

Substituting our expression for $f(t)$:

$$y(t) = \left(-\frac{1}{5} + \frac{1}{5}e^{5t}\right) + u(t-3)\left(-\frac{1}{5} + \frac{1}{5}e^{5(t-3)}\right)$$

The final solution is:

$$y(t) = -\frac{1}{5} + \frac{1}{5}e^{5t} + \frac{1}{5}u(t-3)\left(e^{5(t-3)} - 1\right)$$

Problem 47. Solve the equation:

$$y(t) - \int_0^t e^{-v}\, y(t-v)\, dv = 3$$

Solution 47. The integral $\int_0^t e^{-v}\, y(t-v)\, dv$ is the convolution of e^{-t} and $y(t)$, denoted as $(e^{-t} * y(t))$. We rewrite the equation as:

$$y(t) - (e^{-t} * y(t)) = 3$$

Taking the Laplace transform of both sides and applying the **Convolution Theorem (Entry #19)**:

$$\mathcal{L}\{y(t)\} - \mathcal{L}\{e^{-t}\} \cdot \mathcal{L}\{y(t)\} = \mathcal{L}\{3\}$$

Using **Entry #2** for the exponential ($a = -1$) and **Entry #1** for the constant 3:

$$Y(s) - \left(\frac{1}{s+1}\right)Y(s) = \frac{3}{s}$$

Factor out $Y(s)$ on the left side:

$$Y(s)\left[1 - \frac{1}{s+1}\right] = \frac{3}{s}$$

Simplify the bracketed term by finding a common denominator:

$$Y(s)\left[\frac{(s+1)-1}{s+1}\right] = \frac{3}{s} \Longrightarrow Y(s)\left[\frac{s}{s+1}\right] = \frac{3}{s}$$

Solve for $Y(s)$ by multiplying both sides by $\frac{s+1}{s}$:

$$Y(s) = \frac{3}{s} \cdot \frac{s+1}{s} = \frac{3s+3}{s^2}$$

Split the fraction into two parts:

$$Y(s) = \frac{3s}{s^2} + \frac{3}{s^2} = \frac{3}{s} + \frac{3}{s^2}$$

Now, take the inverse Laplace transform using **Entry #1** for $\frac{1}{s}$ and **Entry #3** for $\frac{1}{s^2}$ $(n = 1)$:

$$y(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

The final solution is:

$$y(t) = 3 + 3t$$

Problem 48. Solve the initial value problem:

$$y'' + y = 4\delta(t - 2\pi), \quad y(0) = 1, \quad y'(0) = 0$$

Solution 48. First, we take the Laplace transform of both sides of the differential equation. Using **Entry #13** for the second derivative and **Entry #20** for the Dirac delta function:

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = 4e^{-2\pi s}$$

Applying the initial conditions $y(0) = 1$ and $y'(0) = 0$:

$$s^2Y(s) - s + Y(s) = 4e^{-2\pi s}$$

Group the $Y(s)$ terms and move the constant to the right:

$$(s^2 + 1)Y(s) = 4e^{-2\pi s} + s$$

Solve for $Y(s)$:

$$Y(s) = \frac{4e^{-2\pi s}}{s^2 + 1} + \frac{s}{s^2 + 1}$$

Next, we take the inverse Laplace transform.

- For $\frac{s}{s^2+1}$, we use **Entry #5** (with $b = 1$): $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$.
- For $4e^{-2\pi s}\frac{1}{s^2+1}$, we use **Entry #4** for the sine part ($\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$) and the time-shifting property from **Entry #16**: $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$.

Combining these:

$$y(t) = 4\sin(t - 2\pi)u(t - 2\pi) + \cos(t)$$

Using the periodic property of the sine function, $\sin(t - 2\pi) = \sin(t)$, we can simplify the final expression:

$$y(t) = 4\sin(t)u(t - 2\pi) + \cos(t)$$

Problem 49. Use the Laplace transform method to solve the initial value problem:

$$y'' - 2y' - 3y = 8e^t, \quad y(0) = 0, \quad y'(0) = 0$$

Solution 49. First, we take the Laplace transform of both sides of the equation. Using **Entry #12** and **Entry #13** for the derivatives and **Entry #2** for the exponential $8e^t$:

$$\mathcal{L}\{y'' - 2y' - 3y\} = \mathcal{L}\{8e^t\}$$

Substituting the formulas and applying the initial conditions $y(0) = 0$ and $y'(0) = 0$:

$$[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] - 3Y(s) = \frac{8}{s - 1}$$

$$(s^2 - 2s - 3)Y(s) = \frac{8}{s - 1}$$

Factoring the quadratic $s^2 - 2s - 3 = (s - 3)(s + 1)$, we solve for $Y(s)$:

$$Y(s) = \frac{8}{(s - 1)(s + 1)(s - 3)}$$

We use Partial Fraction Decomposition (PFD) to split the expression:

$$\frac{8}{(s - 1)(s + 1)(s - 3)} = \frac{A}{s - 1} + \frac{B}{s + 1} + \frac{C}{s - 3}$$

Multiplying by the common denominator:

$$8 = A(s + 1)(s - 3) + B(s - 1)(s - 3) + C(s - 1)(s + 1)$$

- **Set** $s = 1$**:** $8 = A(2)(-2) \Longrightarrow 8 = -4A \Longrightarrow A = -2$
- **Set** $s = -1$**:** $8 = B(-2)(-4) \Longrightarrow 8 = 8B \Longrightarrow B = 1$
- **Set** $s = 3$**:** $8 = C(2)(4) \Longrightarrow 8 = 8C \Longrightarrow C = 1$

Substituting these values back into $Y(s)$:

$$Y(s) = \frac{-2}{s - 1} + \frac{1}{s + 1} + \frac{1}{s - 3}$$

Now, we take the inverse Laplace transform using **Entry #2** ($e^{at} \leftrightarrow \frac{1}{s-a}$):

$$y(t) = -2\mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s - 3}\right\}$$

The final solution is:

$$y(t) = -2e^{t} + e^{-t} + e^{3t}$$

Problem 50. Solve the initial value problem:

$$y'' + 2y' + y = \delta(t-3), \quad y(0) = 1, \quad y'(0) = -1$$

Solution 50. First, we take the Laplace transform of both sides of the differential equation. Using **Entry #12**, **Entry #13**, and **Entry #20** from the table:

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + Y(s) = e^{-3s}$$

Applying the initial conditions $y(0) = 1$ and $y'(0) = -1$:

$$[s^2Y(s) - s(1) - (-1)] + 2[sY(s) - 1] + Y(s) = e^{-3s}$$

Simplify and group the $Y(s)$ terms:

$$s^2Y(s) - s + 1 + 2sY(s) - 2 + Y(s) = e^{-3s}$$

$$(s^2 + 2s + 1)Y(s) - s - 1 = e^{-3s}$$

We recognize the quadratic as a perfect square: $s^2 + 2s + 1 = (s+1)^2$. Substituting this in:

$$(s+1)^2Y(s) = s + 1 + e^{-3s}$$

Solve for $Y(s)$:

$$Y(s) = \frac{s+1}{(s+1)^2} + \frac{e^{-3s}}{(s+1)^2} = \frac{1}{s+1} + e^{-3s}\frac{1}{(s+1)^2}$$

Next, we take the inverse Laplace transform.

- For $\frac{1}{s+1}$, we use **Entry #2** ($a = -1$): $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$.
- For $e^{-3s}\frac{1}{(s+1)^2}$, we define $F(s) = \frac{1}{(s+1)^2}$. From **Entry #8** (with $n = 1, a = -1$), we have $f(t) = te^{-t}$.
- We apply the time-shifting property from **Entry #16**, $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$, with $a = 3$.

Combining these results:

$$y(t) = e^{-t} + f(t-3)u(t-3)$$

Substituting $f(t-3) = (t-3)e^{-(t-3)}$:

$$y(t) = e^{-t} + (t-3)e^{-(t-3)}u(t-3)$$

The final solution is:

$$y(t) = e^{-t} + (t-3)e^{-t+3}u(t-3)$$

Problem 51. Find the following information for the function:

$$g(t) = e^{3(t-2)}u(t-2)$$

Determine the values of $g(1)$, $g(3)$, and the Laplace transform $\mathcal{L}\{g(t)\}$.

Solution 51. The function $g(t)$ involves the unit step function $u(t-2)$, which is defined as:

$$u(t-2) = \begin{cases} 0 & t < 2 \\ 1 & t \geq 2 \end{cases}$$

1. Finding $g(1)$: Since $t = 1$ is less than 2, the unit step function $u(1-2) = 0$.

$$g(1) = e^{3(1-2)} \cdot 0 = 0$$

2. Finding $g(3)$: Since $t = 3$ is greater than 2, the unit step function $u(3-2) = 1$.

$$g(3) = e^{3(3-2)} \cdot 1 = e^{3(1)} = e^3$$

3. Finding the Laplace Transform $\mathcal{L}\{g(t)\}$: We use the time-shifting property from **Entry #16** of the table, which states:

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

In our case, $a = 2$ and the unshifted function is $f(t) = e^{3t}$. Using **Entry #2**, the transform of $f(t)$ is:

$$F(s) = \mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$$

Applying the shift e^{-2s}:

$$\mathcal{L}\{g(t)\} = \frac{1}{s-3}e^{-2s}$$

Final Answers: $g(1) = 0$, $g(3) = e^3$, $\mathcal{L}\{g\} = \frac{e^{-2s}}{s-3}$

Problem 52. Solve the initial value problem:

$$y' + 3y = 1 + u(t-2), \quad y(0) = 0$$

Solution 52. We begin by taking the Laplace transform of both sides of the differential equation. Using **Entry #12** for the derivative, **Entry #1** for the constant 1, and **Entry #17** for the unit step function $u(t-2)$:

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{1\} + \mathcal{L}\{u(t-2)\}$$

Substituting the formulas and applying the initial condition $y(0) = 0$:

$$[sY(s) - y(0)] + 3Y(s) = \frac{1}{s} + \frac{1}{s}e^{-2s}$$

$$(s+3)Y(s) = \frac{1}{s} + \frac{e^{-2s}}{s}$$

Solving for $Y(s)$:

$$Y(s) = \frac{1}{s(s+3)} + \frac{1}{s(s+3)}e^{-2s}$$

Next, we perform Partial Fraction Decomposition (PFD) on the term $F(s) = \frac{1}{s(s+3)}$:

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} \Longrightarrow 1 = A(s+3) + Bs$$

- Let $s = 0$: $1 = 3A \Longrightarrow A = 1/3$
- Let $s = -3$: $1 = -3B \Longrightarrow B = -1/3$

So, $F(s) = \frac{1}{3} \cdot \frac{1}{s} - \frac{1}{3} \cdot \frac{1}{s+3}$. Taking the inverse transform of $F(s)$ using **Entry #1** and **Entry #2**:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{3} - \frac{1}{3}e^{-3t}$$

The expression for $Y(s)$ is $Y(s) = F(s) + F(s)e^{-2s}$. Applying the inverse Laplace transform and using the time-shifting property from **Entry #16**:

$$y(t) = f(t) + f(t-2)u(t-2)$$

Substituting our expression for $f(t)$:

$$y(t) = \left(\frac{1}{3} - \frac{1}{3}e^{-3t}\right) + \left(\frac{1}{3} - \frac{1}{3}e^{-3(t-2)}\right)u(t-2)$$

Problem 53. Solve the initial value problem:

$$y'' - 2y' + 5y = 2\delta(t-3), \quad y(0) = 0, \quad y'(0) = 2$$

Solution 53. First, we take the Laplace transform of both sides of the differential equation. Using **Entry #12**, **Entry #13**, and **Entry #20** from the table:

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 2\mathcal{L}\{\delta(t-3)\}$$

Substituting the formulas and applying the initial conditions $y(0) = 0$ and $y'(0) = 2$:

$$[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 5Y(s) = 2e^{-3s}$$

$$[s^2Y(s) - 0 - 2] - 2[sY(s) - 0] + 5Y(s) = 2e^{-3s}$$

Grouping the $Y(s)$ terms:

$$(s^2 - 2s + 5)Y(s) = 2 + 2e^{-3s}$$

Solving for $Y(s)$:

$$Y(s) = \frac{2}{s^2 - 2s + 5} + \frac{2}{s^2 - 2s + 5}e^{-3s}$$

To find the inverse transform, we complete the square in the denominator:

$$s^2 - 2s + 5 = (s^2 - 2s + 1) - 1 + 5 = (s - 1)^2 + 4 = (s - 1)^2 + 2^2$$

Let $F(s) = \frac{2}{(s-1)^2+2^2}$. Using **Entry #9** ($b = 2, a = 1$):

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^t \sin(2t)$$

Applying the inverse transform to $Y(s)$ using the time-shifting property from **Entry #16** ($a = 3$):

$$y(t) = f(t) + f(t - 3)u(t - 3)$$

Substituting our expression for $f(t)$:

$$y(t) = e^t \sin(2t) + e^{t-3}\sin(2(t - 3))u(t - 3)$$

The final solution is:

$$y(t) = e^t \sin(2t) + e^{t-3}\sin(2t - 6)u(t - 3)$$

Problem 54. Solve the integro-differential equation

$$y' + 2y - \int_0^t 2\, y(\tau)\sin(t - \tau)\, d\tau = -\cos t$$

subject to the initial condition $y(0) = 0$.

Solution 54. We identify the integral term as the convolution product $(2y * \sin t)$. Taking the Laplace transform of both sides using the **Convolution Theorem (Entry #19)**:

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} - 2\mathcal{L}\{y * \sin t\} = -\mathcal{L}\{\cos t\}$$

Using **Entry #12** for the derivative, **Entry #4** for sine, and **Entry #5** for cosine:

$$[sY(s) - y(0)] + 2Y(s) - 2Y(s)\left(\frac{1}{s^2 + 1}\right) = -\frac{s}{s^2 + 1}$$

Applying the initial condition $y(0) = 0$:

$$\left(s + 2 - \frac{2}{s^2 + 1}\right)Y(s) = -\frac{s}{s^2 + 1}$$

Find a common denominator for the terms in the parentheses:

$$\left(\frac{(s+2)(s^2+1)-2}{s^2+1}\right)Y(s) = -\frac{s}{s^2+1}$$

Expanding the numerator: $(s+2)(s^2+1)-2 = s^3+s+2s^2+2-2 = s^3+2s^2+s$.

$$\left(\frac{s^3+2s^2+s}{s^2+1}\right)Y(s) = -\frac{s}{s^2+1}$$

Solve for $Y(s)$ by multiplying by the reciprocal:

$$Y(s) = -\frac{s}{s^2+1}\cdot\frac{s^2+1}{s^3+2s^2+s} = -\frac{s}{s(s^2+2s+1)} = -\frac{1}{(s+1)^2}$$

Now, take the inverse Laplace transform. Using **Entry #8** with $n = 1$ and $a = -1$:

$$y(t) = \mathcal{L}^{-1}\left\{-\frac{1}{(s+1)^2}\right\} = -te^{-t}$$

The final solution is:

$$y(t) = -te^{-t}$$

Problem 55. Solve the initial value problem

$$y'' + y = \delta(t-\pi), \quad y(0) = 0, \quad y'(0) = 1$$

and compute the numerical values of $y\left(\frac{\pi}{2}\right)$ and $y\left(\frac{3\pi}{2}\right)$.

Solution 55. We use the Laplace transform method due to the presence of the Dirac delta function. Taking the Laplace transform of both sides:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\}$$

Using **Entry #13** for the derivative and **Entry #20** for the Dirac delta:

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = e^{-\pi s}$$

Applying the initial conditions $y(0) = 0$ and $y'(0) = 1$:

$$s^2Y(s) - 1 + Y(s) = e^{-\pi s}$$

$$(s^2+1)Y(s) = 1 + e^{-\pi s}$$

Solving for $Y(s)$:

$$Y(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

Taking the inverse Laplace transform using **Entry #4** and the shifting property in **Entry #16**:

$$y(t) = \sin(t) + \sin(t - \pi)u(t - \pi)$$

Numerical Evaluations: Recall the definition of the unit step function: $u(t - a) = 1$ if $t \geq a$ and 0 if $t < a$.

1. For $t = \frac{\pi}{2}$: Since $\frac{\pi}{2} < \pi$, $u\left(\frac{\pi}{2} - \pi\right) = 0$.

$$y\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2} - \pi\right) \cdot 0 = 1 + 0 = 1$$

2. For $t = \frac{3\pi}{2}$: Since $\frac{3\pi}{2} > \pi$, $u\left(\frac{3\pi}{2} - \pi\right) = 1$.

$$y\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) + \sin\left(\frac{3\pi}{2} - \pi\right) \cdot 1 = \sin\left(\frac{3\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)$$

$$y\left(\frac{3\pi}{2}\right) = -1 + 1 = 0$$

Final Answer:

$$y(t) = \sin(t) + \sin(t - \pi)u(t - \pi)$$

$$y\left(\frac{\pi}{2}\right) = 1, \quad y\left(\frac{3\pi}{2}\right) = 0$$

Problem 56. Solve the integral-differential equation using the method of Laplace transforms:

$$y'(t) + 2y(t) - 2\int_0^t y\,(v)\sin(t - v)\,dv = \cos(t), \quad y(0) = 0$$

Solution 56. We identify the integral term as a convolution product $(y * \sin t)$. Taking the Laplace transform of both sides and using the **Convolution Theorem (Entry #19)**:

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} - 2\mathcal{L}\{y * \sin t\} = \mathcal{L}\{\cos t\}$$

Using **Entry #12** for the derivative, **Entry #4** for sine, and **Entry #5** for cosine:

$$[sY(s) - y(0)] + 2Y(s) - 2Y(s)\left(\frac{1}{s^2 + 1}\right) = \frac{s}{s^2 + 1}$$

Applying the initial condition $y(0) = 0$:

$$\left(s + 2 - \frac{2}{s^2 + 1}\right)Y(s) = \frac{s}{s^2 + 1}$$

We find a common denominator for the terms in the parentheses:

$$\left(\frac{(s + 2)(s^2 + 1) - 2}{s^2 + 1}\right)Y(s) = \frac{s}{s^2 + 1}$$

Expanding the numerator: $(s+2)(s^2+1)-2 = s^3+s+2s^2+2-2 = s^3+2s^2+s$.

$$\left(\frac{s^3+2s^2+s}{s^2+1}\right)Y(s) = \frac{s}{s^2+1}$$

Solving for $Y(s)$:

$$Y(s) = \frac{s}{s^2+1}\cdot\frac{s^2+1}{s^3+2s^2+s} = \frac{s}{s(s^2+2s+1)} = \frac{1}{(s+1)^2}$$

Now, take the inverse Laplace transform. Using **Entry #8** with $n=1$ and $a=-1$:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$$

The final solution is:

$$y(t) = te^{-t}$$

Problem 57. Solve the initial value problem using the method of Laplace transforms:

$$y''(t) - 4y'(t) + 5y(t) = \delta(t-3), \quad y(0)=0, \quad y'(0)=1$$

Solution 57. We begin by taking the Laplace transform of both sides of the differential equation. Using **Entry #12**, **Entry #13**, and **Entry #20** from the table:

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-3)\}$$

Substituting the formulas and applying the initial conditions $y(0)=0$ and $y'(0)=1$:

$$[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + 5Y(s) = e^{-3s}$$

$$[s^2Y(s) - 0 - 1] - 4[sY(s) - 0] + 5Y(s) = e^{-3s}$$

Grouping the $Y(s)$ terms:

$$(s^2-4s+5)Y(s) = 1 + e^{-3s}$$

Solving for $Y(s)$:

$$Y(s) = \frac{1}{s^2-4s+5} + \frac{e^{-3s}}{s^2-4s+5}$$

To find the inverse transform, we complete the square in the denominator:

$$s^2-4s+5 = (s^2-4s+4)+1 = (s-2)^2+1^2$$

Let $F(s) = \frac{1}{(s-2)^2+1}$. Using **Entry #9** with $b=1$ and $a=2$:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{2t}\sin(t)$$

Now we apply the inverse transform to $Y(s)$. For the second term, we use the time-shifting property from **Entry #16** with $a = 3$:

$$y(t) = f(t) + f(t-3)u(t-3)$$

Substituting our expression for $f(t)$:

$$y(t) = e^{2t}\sin(t) + e^{2(t-3)}\sin(t-3)u(t-3)$$

The final solution is:

$$y(t) = e^{2t}\sin(t) + e^{2t-6}\sin(t-3)u(t-3)$$

Problem 58. Solve the initial value problem using the method of Laplace transforms:

$$y'' + 5y' + 6y = t\delta(t-3), \quad y(0) = 0, \quad y'(0) = 0$$

Solution 58. First, we take the Laplace transform of both sides of the equation. For the right-hand side, we use the property $\mathcal{L}\{g(t)\delta(t-a)\} = g(a)e^{-as}$ from the definition of the transform:

$$\mathcal{L}\{y'' + 5y' + 6y\} = \mathcal{L}\{t\delta(t-3)\} = 3e^{-3s}$$

Using **Entry #12** and **Entry #13** for the derivatives and applying the initial conditions $y(0) = 0$ and $y'(0) = 0$:

$$[s^2Y(s) - sy(0) - y'(0)] + 5[sY(s) - y(0)] + 6Y(s) = 3e^{-3s}$$

$$(s^2 + 5s + 6)Y(s) = 3e^{-3s}$$

Solving for $Y(s)$:

$$Y(s) = \frac{3}{s^2 + 5s + 6}e^{-3s} = \frac{3}{(s+3)(s+2)}e^{-3s}$$

We perform Partial Fraction Decomposition (PFD) on $F(s) = \frac{3}{(s+3)(s+2)}$:

$$\frac{3}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2} \Longrightarrow 3 = A(s+2) + B(s+3)$$

- Let $s = -2$: $3 = B(1) \Longrightarrow B = 3$
- Let $s = -3$: $3 = A(-1) \Longrightarrow A = -3$

So, $F(s) = \frac{3}{s+2} - \frac{3}{s+3}$. Taking the inverse transform using **Entry #2**:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 3e^{-2t} - 3e^{-3t}$$

Now we apply the time-shifting property from **Entry #16** to $Y(s) = F(s)e^{-3s}$ with $a = 3$:

$$y(t) = f(t-3)u(t-3)$$

Substituting our expression for $f(t-3)$:

$$y(t) = 3\left(e^{-2(t-3)} - e^{-3(t-3)}\right)u(t-3)$$

The final solution is:

$$y(t) = \begin{cases} 0 & t < 3 \\ 3e^{-2t+6} - 3e^{-3t+9} & t \geq 3 \end{cases}$$

Problem 59. Solve the integral equation using the method of Laplace transforms:

$$(\dagger) \quad y(t) + \int_0^t (t-v)y(v)\,dv = 2t$$

Solution 59. The integral term $\int_0^t (t-v)y(v)\,dv$ is the convolution product of $f(t) = t$ and $g(t) = y(t)$, denoted as $(t * y)(t)$. We can rewrite the equation as:

$$y(t) + (t * y)(t) = 2t$$

Taking the Laplace transform of both sides and applying the **Convolution Theorem (Entry #19)**:

$$\mathcal{L}\{y\} + \mathcal{L}\{t\} \cdot \mathcal{L}\{y\} = \mathcal{L}\{2t\}$$

Using **Entry #3** for t (where $n = 1$):

$$Y(s) + \frac{1}{s^2}Y(s) = \frac{2}{s^2}$$

Factor out $Y(s)$ on the left side:

$$Y(s)\left(1 + \frac{1}{s^2}\right) = \frac{2}{s^2}$$

Simplify the bracketed term by finding a common denominator:

$$Y(s)\left(\frac{s^2+1}{s^2}\right) = \frac{2}{s^2}$$

Solve for $Y(s)$ by multiplying both sides by $\frac{s^2}{s^2+1}$:

$$Y(s) = \frac{2}{s^2} \cdot \frac{s^2}{s^2+1} = \frac{2}{s^2+1}$$

Now, take the inverse Laplace transform. Using **Entry #4** for $\sin(bt)$ (where $b = 1$):

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\} = 2\sin(t)$$

The final solution is:

$$y(t) = 2\sin(t)$$

Problem 60. Solve the initial value problem using the method of Laplace transforms:

$$y'' + 2y' + 2y = g(t), \quad y(0) = y'(0) = 0$$

where $g(t) = 0$ for $t \leq 2$ and $g(t) = 1$ for $t > 2$.

Solution 60. First, we express the piecewise function $g(t)$ using the unit step function $u(t-a)$. Based on the definition provided:

$$g(t) = u(t-2)$$

Next, we take the Laplace transform of both sides of the differential equation. Let $\mathcal{L}\{y(t)\} = Y(s)$. Using **Entry #13** for the second derivative, **Entry #12** for the first derivative, and **Entry #17** for the unit step function:

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2Y(s) = \frac{e^{-2s}}{s}$$

Applying the initial conditions $y(0) = 0$ and $y'(0) = 0$:

$$(s^2 + 2s + 2)Y(s) = \frac{e^{-2s}}{s}$$

Solving for $Y(s)$:

$$Y(s) = e^{-2s} \cdot \frac{1}{s(s^2 + 2s + 2)}$$

Let $F(s) = \frac{1}{s(s^2+2s+2)}$. We use partial fraction decomposition to simplify $F(s)$:

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$

Multiplying by the denominator gives:

$$1 = A(s^2 + 2s + 2) + (Bs + C)s = (A + B)s^2 + (2A + C)s + 2A$$

Matching the coefficients:

- $2A = 1 \Longrightarrow A = \frac{1}{2}$
- $2A + C = 0 \Longrightarrow 1 + C = 0 \Longrightarrow C = -1$
- $A + B = 0 \Longrightarrow \frac{1}{2} + B = 0 \Longrightarrow B = -\frac{1}{2}$

So, $F(s) = \frac{1}{2} \cdot \frac{1}{s} + \frac{-\frac{1}{2}s-1}{s^2+2s+2}$. We complete the square in the denominator: $s^2 + 2s + 2 = (s + 1)^2 + 1$. To match the table entries, we rewrite the numerator in terms of $(s + 1)$:

$$-\frac{1}{2}s - 1 = -\frac{1}{2}(s+1) - \frac{1}{2}$$

Substituting this back into $F(s)$:

$$F(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s+1}{(s+1)^2 + 1^2}\right) - \frac{1}{2}\left(\frac{1}{(s+1)^2 + 1^2}\right)$$

Taking the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ using **Entries #1, #10, and #9**:

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos(t) - \frac{1}{2}e^{-t}\sin(t)$$

Finally, we find $y(t) = \mathcal{L}^{-1}\{e^{-2s}F(s)\}$ using the shifting property in **Entry #16**:

$$y(t) = f(t-2)u(t-2)$$

The final solution is:

$$y(t) = \left[\frac{1}{2} - \frac{1}{2}e^{-(t-2)}\cos(t-2) - \frac{1}{2}e^{-(t-2)}\sin(t-2)\right]u(t-2)$$

9.1/5.1 - Introductions to Systems of Linear Differential Equations, Interconnected Fluid Tanks

Problem 61. Two tanks initially hold 200 gallons of pure water each. A mixture with concentration 5 oz/gal flows into Tank 1 at 5 gal/min. The mixture drains from Tank 1 into Tank 2 at 2 gal/min and into the environment at 1 gal/min. Tank 2 drains into the environment at 2 gal/min. Let $Q_1(t)$ and $Q_2(t)$ denote the salt (oz) in Tanks 1 and 2. Set up, but do not solve, a system of differential equations with initial conditions modeling the process.

Solution 61. For Tank 1, the volume is

$$V_1(t) = 200 + 5t - 2t - 1t = 200 + 2t.$$

For Tank 2, the volume is

$$V_2(t) = 200 + 2t - 2t = 200.$$

The rate of salt entering Tank 1 is

$$5 \cdot 5 = 25.$$

The concentration in Tank 1 is

$$\frac{Q_1(t)}{200 + 2t}.$$

Thus the rate of salt leaving Tank 1 is

$$(2+1)\frac{Q_1(t)}{200+2t} = \frac{3Q_1(t)}{200+2t}.$$

Therefore,

$$\frac{dQ_1}{dt} = 25 - \frac{3Q_1}{200+2t}.$$

For Tank 2, salt enters from Tank 1 at rate

$$2\frac{Q_1(t)}{200+2t}.$$

The concentration in Tank 2 is

$$\frac{Q_2(t)}{200}.$$

Thus salt leaves Tank 2 at rate

$$2\frac{Q_2(t)}{200} = \frac{Q_2(t)}{100}.$$

Therefore, the system is

$$\begin{cases} Q_1' = 25 - \dfrac{3Q_1}{200+2t}, \\ Q_2' = \dfrac{2Q_1}{200+2t} - \dfrac{Q_2}{100}, \end{cases}$$

with initial conditions

$$Q_1(0) = 0, \qquad Q_2(0) = 0.$$

Problem 62. Consider two interconnected tanks. Tank 1 initially contains 40 grams of sugar dissolved in 60 liters of water, and Tank 2 initially contains 50 liters of water with 25 grams of sugar. Water containing 2 grams of sugar per liter flows into Tank 2 at 3 liters per minute. The mixture drains from Tank 2 at 4 liters per minute, with 1.5 liters per minute flowing into Tank 1 and the rest leaving the system. The mixture in Tank 1 flows back into Tank 2 at 1.5 liters per minute. If $S_1(t)$ and $S_2(t)$ denote the amounts of sugar in Tanks 1 and 2, respectively, set up, but do not solve, an initial value problem modeling the process.

Solution 62. For Tank 1, the volume is constant because the inflow and outflow rates are both 1.5 liters per minute. Thus

$$V_1(t) = 60.$$

For Tank 2, the volume is also constant because the total inflow and outflow rates are both 4.5 liters per minute. Thus

$$V_2(t) = 50.$$

For Tank 1, sugar enters from Tank 2 at rate

$$1.5\frac{S_2(t)}{50} = \frac{3S_2(t)}{100}.$$

Sugar leaves Tank 1 at rate

$$1.5\frac{S_1(t)}{60} = \frac{S_1(t)}{40}.$$

Thus

$$S_1' = \frac{3S_2}{100} - \frac{S_1}{40}.$$

For Tank 2, sugar enters from the outside at rate

$$3 \cdot 2 = 6.$$

Sugar also enters from Tank 1 at rate

$$1.5\frac{S_1(t)}{60} = \frac{S_1(t)}{40}.$$

Sugar leaves Tank 2 at rate

$$4\frac{S_2(t)}{50} = \frac{2S_2(t)}{25}.$$

Thus

$$S_2' = 6 + \frac{S_1}{40} - \frac{2S_2}{25}.$$

Therefore, the initial value problem is

$$\begin{cases} S_1' = \frac{3S_2}{100} - \frac{S_1}{40}, \\ S_2' = 6 + \frac{S_1}{40} - \frac{2S_2}{25}, \end{cases}$$

with initial conditions

$$S_1(0) = 40, \qquad S_2(0) = 25.$$

Problem 63. Two very large tanks initially hold 100 gallons of pure water each. A mixture with concentration 3 oz/gal flows into Tank 1 at 5 gal/min. The mixture drains from Tank 1 into the environment at 2 gal/min and into Tank 2 at 3 gal/min. Another mixture with concentration 4 oz/gal flows into Tank 2 at 2 gal/min. Tank 2 drains into the environment at 2 gal/min. If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in Tanks 1 and 2, respectively, set up, but do not solve, the differential equations and initial conditions modeling the process.

Solution 63. For Tank 1, the volume is constant because the inflow and outflow rates are both 5 gallons per minute. Thus

$$V_1(t) = 100.$$

For Tank 2, the volume is increasing because the total inflow rate is $3 + 2 = 5$ gallons per minute and the outflow rate is 2 gallons per minute. Thus

$$V_2(t) = 100 + 3t.$$

For Tank 1, salt enters from the outside at rate

$$5 \cdot 3 = 15.$$

Salt leaves Tank 1 at rate

$$(2+3)\frac{Q_1(t)}{100} = \frac{Q_1(t)}{20}.$$

Thus

$$Q_1' = 15 - \frac{Q_1}{20}.$$

For Tank 2, salt enters from Tank 1 at rate

$$3\frac{Q_1(t)}{100} = \frac{3Q_1(t)}{100}.$$

Salt also enters from the outside at rate

$$2 \cdot 4 = 8.$$

Salt leaves Tank 2 at rate

$$2\frac{Q_2(t)}{100+3t}.$$

Thus

$$Q_2' = 8 + \frac{3Q_1}{100} - \frac{2Q_2}{100+3t}.$$

Therefore, the initial value problem is

$$\begin{cases} Q_1' = 15 - \dfrac{Q_1}{20}, \\ Q_2' = 8 + \dfrac{3Q_1}{100} - \dfrac{2Q_2}{100+3t}, \end{cases}$$

with initial conditions

$$Q_1(0) = 0, \qquad Q_2(0) = 0.$$

Problem 64. Consider a system of two interconnected tanks. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 5 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

(a) Set up, but do not solve, an initial value problem modeling the amount of salt in Tank 1 and Tank 2, respectively, at any time t.

(b) For which times t is your model in part (a) valid? Explain why this is so.

Solution 64. Let $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in Tank 1 and Tank 2, respectively. For Tank 1, the volume is

$$V_1(t) = 30 + 1.5t + 1.5t - 3t = 30.$$

For Tank 2, the volume is

$$V_2(t) = 20 + 3t + 1t - 5t = 20 - t.$$

For Tank 1, salt enters from the outside at rate

$$1.5 \cdot 1 = \frac{3}{2}.$$

Salt also enters from Tank 2 at rate

$$1.5\frac{Q_2(t)}{20 - t}.$$

Salt leaves Tank 1 at rate

$$3\frac{Q_1(t)}{30} = \frac{Q_1(t)}{10}.$$

Thus

$$Q_1' = \frac{3}{2} + \frac{3Q_2}{2(20 - t)} - \frac{Q_1}{10}.$$

For Tank 2, salt enters from Tank 1 at rate

$$3\frac{Q_1(t)}{30} = \frac{Q_1(t)}{10}.$$

Salt also enters from the outside at rate

$$1 \cdot 3 = 3.$$

Salt leaves Tank 2 at rate

$$5\frac{Q_2(t)}{20-t}.$$

Thus

$$Q_2' = 3 + \frac{Q_1}{10} - \frac{5Q_2}{20-t}.$$

Therefore, the initial value problem is

$$\begin{cases} Q_1' = \dfrac{3}{2} + \dfrac{3Q_2}{2(20-t)} - \dfrac{Q_1}{10}, \\ Q_2' = 3 + \dfrac{Q_1}{10} - \dfrac{5Q_2}{20-t}, \end{cases}$$

with initial conditions

$$Q_1(0) = 25, \qquad Q_2(0) = 15.$$

The model is valid for

$$0 \le t < 20,$$

because

$$V_2(t) = 20 - t,$$

so Tank 2 becomes empty at $t = 20$. After that, the concentration $\frac{Q_2(t)}{20-t}$ is no longer defined.

Problem 65. Consider two tanks holding initially 100 gallons of pure water each. A mixture of salt and water at a concentration of 4 ounces per gallon flows into Tank 1 at a rate of 5 gallons per minute. The well-stirred mixture in Tank 1 drains into the environment at a rate of 3 gallons per minute, and from Tank 1 into Tank 2 at a rate of 2 gallons per minute. Another mixture of salt and water at a concentration of 5 ounces per gallon flows into Tank 2 at a rate of 2 gallons per minute. The well-stirred mixture in Tank 2 drains into the environment at a rate of 2 gallons per minute.

If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in ounces at time t in Tanks 1 and 2, respectively, SET UP, BUT DO NOT SOLVE, the differential equations and initial conditions that model the flow process.

Solution 65. The net flow into tank 1 is 0. We have 5 gallons going per minute into tank 1 and 5 gallons leaving the tank. So $V_1(t) = V_0 = 100$
The net flow into tank 2 is $+2$. There is 4 gallons going into tank 2 and only 2 gallons leaving the tank. The volume in tank 2 at time t is given by $V_2(t) = v_2(0) + \text{net flow}t = 100 + 2t$
The Net rate is given by: Net rate = rate in - rate out.
Thus,

$$\frac{dQ_1}{dt} = \left(\frac{4oz}{gal}\right)\left(\frac{5gal}{min}\right) - \left(\frac{Q_1(t)oz}{V_1(t)gal}\right)\left(\frac{5gal}{min}\right)$$
$$\frac{dQ_2}{dt} = \left(\frac{5oz}{gal}\right)\left(\frac{2gal}{min}\right) + \left(\frac{Q_1(t)oz}{V_1(t)gal}\right)\left(\frac{2gal}{min}\right) - \left(\frac{Q_2(t)oz}{V_2(t)gal}\right)\left(\frac{2gal}{min}\right)$$

This leads to:

$$Q_1' = 20 - \frac{Q_1}{20}, \qquad Q_1(0) = 0$$
$$Q_2' = 10 + \frac{Q_1}{50} - \frac{Q_2}{50+t}, \quad Q_2(0) = 0$$

Problem 66. Consider two interconnected tanks. Tank 1 initially contains 25 gal of water and 20 oz of salt, and Tank 2 initially contains 15 gal of water and 10 oz of salt. Water containing 2 oz/gal of salt flows into Tank 1 at a rate of 2.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 4 oz/gal of salt also flows into Tank 2 at a rate of 2 gal/min from outside. The mixture drains from Tank 2 at a rate of 5 gal/min, of which some flows back into Tank 1 at a rate of 2.5 gal/min, while the remainder leaves the system.
Set up, BUT DO NOT SOLVE, an initial value problem which models the amount of salt in each tank at all future times.

Solution 66. The net flow into tank 1 is $+2$. We have 5 gallons per minute going into Tank 1 and 3 gallons per minute leaving Tank 1. So

$$V_1(t) = 25 + 2t.$$

The net flow into tank 2 is 0. We have 5 gallons per minute going into Tank 2 and 5 gallons per minute leaving Tank 2. So

$$V_2(t) = 15.$$

The Net rate is given by: Net rate = rate in - rate out.

Thus,

$$\frac{dQ_1}{dt} = \left(\frac{2\text{ oz}}{\text{gal}}\right)\left(\frac{2.5\text{ gal}}{\text{min}}\right) + \left(\frac{Q_2(t)\text{ oz}}{V_2(t)\text{ gal}}\right)\left(\frac{2.5\text{ gal}}{\text{min}}\right) - \left(\frac{Q_1(t)\text{ oz}}{V_1(t)\text{ gal}}\right)\left(\frac{3\text{ gal}}{\text{min}}\right)$$
$$\frac{dQ_2}{dt} = \left(\frac{4\text{ oz}}{\text{gal}}\right)\left(\frac{2\text{ gal}}{\text{min}}\right) + \left(\frac{Q_1(t)\text{ oz}}{V_1(t)\text{ gal}}\right)\left(\frac{3\text{ gal}}{\text{min}}\right) - \left(\frac{Q_2(t)\text{ oz}}{V_2(t)\text{ gal}}\right)\left(\frac{5\text{ gal}}{\text{min}}\right).$$

This leads to:

$$Q_1' = 5 + \frac{Q_2}{6} - \frac{3Q_1}{25+2t}, \qquad Q_1(0) = 20,$$
$$Q_2' = 8 + \frac{3Q_1}{25+2t} - \frac{Q_2}{3}, \qquad Q_2(0) = 10.$$

Problem 67. Consider a system of two interconnected tanks. Tank 1 contains initially 20 gallons of pure water, and Tank 2 initially contains 50 gallons of water and 20 oz of salt. A mixture of salt and water at a concentration of 5 ounces per gallon flows into Tank 1 at a rate of 2 gallons per minute. The well-stirred mixture in Tank 1 drains into Tank 2 at a rate of 3 gallons per minute. A mixture of salt and water at a concentration of 1 ounce per gallon flows into Tank 2 at a rate of 4 gallons per minute, and the well-stirred mixture in Tank 2 drains at a rate of 7 gallons per minute, of which some flows back into Tank 1 at a rate of 2 gallons per minute, while the remainder leaves the system.
If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in ounces at time t in Tanks 1 and 2, respectively, SET UP, BUT DO NOT SOLVE, an initial value problem modeling the amount of salt in Tank 1 and Tank 2 at any time t .

Solution 67. The net flow into tank 1 is $+1$. We have 4 gallons per minute going into Tank 1 and 3 gallons per minute leaving Tank 1. So

$$V_1(t) = 20 + t.$$

The net flow into tank 2 is 0. We have 7 gallons per minute going into Tank 2 and 7 gallons per minute leaving Tank 2. So

$$V_2(t) = 50.$$

The Net rate is given by: Net rate = rate in - rate out.

Thus,

$$\begin{aligned}\frac{dQ_1}{dt} &= \left(\frac{5\text{ oz}}{\text{gal}}\right)\left(\frac{2\text{ gal}}{\text{min}}\right) + \left(\frac{Q_2(t)\text{ oz}}{V_2(t)\text{ gal}}\right)\left(\frac{2\text{ gal}}{\text{min}}\right) - \left(\frac{Q_1(t)\text{ oz}}{V_1(t)\text{ gal}}\right)\left(\frac{3\text{ gal}}{\text{min}}\right)\\ \frac{dQ_2}{dt} &= \left(\frac{1\text{ oz}}{\text{gal}}\right)\left(\frac{4\text{ gal}}{\text{min}}\right) + \left(\frac{Q_1(t)\text{ oz}}{V_1(t)\text{ gal}}\right)\left(\frac{3\text{ gal}}{\text{min}}\right) - \left(\frac{Q_2(t)\text{ oz}}{V_2(t)\text{ gal}}\right)\left(\frac{7\text{ gal}}{\text{min}}\right).\end{aligned}$$

This leads to:

$$\begin{aligned}Q_1' &= 10 + \frac{2Q_2}{50} - \frac{3Q_1}{20+t}, \qquad Q_1(0) = 0,\\ Q_2' &= 4 + \frac{3Q_1}{20+t} - \frac{7Q_2}{50}, \qquad Q_2(0) = 20.\end{aligned}$$

Problem 68. Two interconnected tanks are each initially filled with 100 gallons of pure water. A mixture containing 2 pounds of salt per gallon of water is pumped into Tank 1 at a rate of 3 gal/min, and into Tank 2 at a rate of 2 gal/min. The well-stirred mixture in Tank 1 flows through a pipe into Tank 2 at a rate of 4 gal/min. Also, the well-stirred mixture in Tank 2 flows through another pipe into Tank 1 at a rate of 5 gal/min and into the environment at a rate of 1 gal/min.

1. Draw and label a simple diagram of the system.

2. Find the volume of each tank at any time $t > 0$ (in minutes).

3. With Q_1 and Q_2 denoting the salt content in tanks 1 and 2, in pounds, set up, but DO NOT SOLVE, the differential equations modeling the system.

Solution 68. (a) To draw the diagram:

- Start with two tanks, tank 1 and tank 2.
- From outside system: Into tank 1, flows in 3 gallons per minute at 2 lbs per gallon. Into tank 2, flows in at 2 gallons per minute at 2 lbs per gallon.
- Inner tank flow. From tank 1 into tank 2 we have a flow of 4 gallons per minute: From tank 2 into tank 2 we have flow of 5 gallons per minute
- Leaving system: Flow from tank 2 leaves the system at 1 gallon per minute

(b) The net flow into tank 1 is $+4$. We have 8 gallons per minute going into Tank 1 and 4 gallons per minute leaving Tank 1. So

$$V_1(t) = 100 + 4t.$$

The net flow into tank 2 is $+1$. We have 6 gallons per minute going into Tank 2 and 5 gallons per minute leaving Tank 2. So

$$V_2(t) = 100 + t.$$

(c) The Net rate is given by: Net rate = rate in - rate out.

Thus,

$$\begin{aligned}
\frac{dQ_1}{dt} &= \left(\frac{2\text{ lb}}{\text{gal}}\right)\left(\frac{3\text{ gal}}{\text{min}}\right) + \left(\frac{Q_2(t)\text{ lb}}{V_2(t)\text{ gal}}\right)\left(\frac{5\text{ gal}}{\text{min}}\right) - \left(\frac{Q_1(t)\text{ lb}}{V_1(t)\text{ gal}}\right)\left(\frac{4\text{ gal}}{\text{min}}\right)\\
\frac{dQ_2}{dt} &= \left(\frac{2\text{ lb}}{\text{gal}}\right)\left(\frac{2\text{ gal}}{\text{min}}\right) + \left(\frac{Q_1(t)\text{ lb}}{V_1(t)\text{ gal}}\right)\left(\frac{4\text{ gal}}{\text{min}}\right) - \left(\frac{Q_2(t)\text{ lb}}{V_2(t)\text{ gal}}\right)\left(\frac{6\text{ gal}}{\text{min}}\right).
\end{aligned}$$

This leads to:

$$\begin{aligned}
Q_1' &= 6 + \frac{5Q_2}{100+t} - \frac{4Q_1}{100+4t}, \qquad Q_1(0) = 0,\\
Q_2' &= 4 + \frac{4Q_1}{100+4t} - \frac{6Q_2}{100+t}, \qquad Q_2(0) = 0.
\end{aligned}$$

Problem 69. Consider an interconnected system of two tanks, Tank 1 and Tank 2.

- Tank 1 initially contains 1 gal of water and 2 oz of salt.
- Tank 2 initially contains 1 gal of water and no salt.

The mixture flows from Tank 1 to Tank 2 at a rate of 1 gal/min. The mixture flows from Tank 2 to Tank 1 at a rate of 1 gal/min. Let $x(t)$ and $y(t)$ be the amount of salt (in oz) in Tank 1 and Tank 2, respectively, at time t.

Set up a system of differential equations, including initial conditions, which models the flow process.

Solution 69. To model the amount of salt in each tank, we use the principle that the rate of change is equal to the rate of salt flowing in minus the rate of salt flowing out:

$$\frac{dA}{dt} = \text{Rate In} - \text{Rate Out}$$

The rate of salt flow is calculated as (Concentration) × (Flow Rate). Since both tanks maintain a constant volume of 1 gallon, the concentrations are $\frac{x}{1}$ for Tank 1 and $\frac{y}{1}$ for Tank 2.

For Tank 1 (x):

- **Rate In:** Salt comes from Tank 2 at 1 gal/min.

$$\text{Rate In} = \left(\frac{y\ \text{oz}}{1\ \text{gal}}\right)\left(1\,\frac{\text{gal}}{\text{min}}\right) = y$$

- **Rate Out:** Salt leaves to Tank 2 at 1 gal/min.

$$\text{Rate Out} = \left(\frac{x\ \text{oz}}{1\ \text{gal}}\right)\left(1\,\frac{\text{gal}}{\text{min}}\right) = x$$

Thus, $\frac{dx}{dt} = y - x$.

For Tank 2 (y):

- **Rate In:** Salt comes from Tank 1 at 1 gal/min.

$$\text{Rate In} = \left(\frac{x\ \text{oz}}{1\ \text{gal}}\right)\left(1\,\frac{\text{gal}}{\text{min}}\right) = x$$

- **Rate Out:** Salt leaves to Tank 1 at 1 gal/min.

$$\text{Rate Out} = \left(\frac{y\ \text{oz}}{1\ \text{gal}}\right)\left(1\,\frac{\text{gal}}{\text{min}}\right) = y$$

Thus, $\frac{dy}{dt} = x - y$.

Initial Conditions: Tank 1 starts with 2 oz of salt, and Tank 2 starts with 0 oz.

$$x(0) = 2, \quad y(0) = 0$$

Final System:

$$\begin{cases} \dfrac{dx}{dt} = y - x \\ \dfrac{dy}{dt} = x - y \end{cases} \quad \text{with} \quad x(0) = 2,\ y(0) = 0$$

9.2-9.3 - Systems and Linear Algebra Problems

Problem 70. If

$$A(t) = \begin{pmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{pmatrix},$$

compute

$$\frac{d}{dt}(A^{-1}(t)).$$

Solution 70. First compute the determinant:

$$\det(A(t)) = \sin^2 t + \cos^2 t = 1.$$

Thus

$$A^{-1}(t) = \begin{pmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{pmatrix}.$$

Differentiating entry-by-entry gives

$$\frac{d}{dt}(A^{-1}(t)) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Problem 71. If

$$A(t) = \begin{pmatrix} 2e^{2t} & e^{-t} \\ 2e^{t} & 3e^{-2t} \end{pmatrix},$$

find

$$\frac{d}{dt}(A^{-1}(t)).$$

Solution 71. First compute the determinant:

$$\det(A(t)) = 2e^{2t} \cdot 3e^{-2t} - e^{-t} \cdot 2e^{t} = 6 - 2 = 4.$$

Thus

$$A^{-1}(t) = \frac{1}{4}\begin{pmatrix} 3e^{-2t} & -e^{-t} \\ -2e^{t} & 2e^{2t} \end{pmatrix}.$$

Differentiating entry-by-entry gives

$$\frac{d}{dt}(A^{-1}(t)) = \frac{1}{4}\begin{pmatrix} -6e^{-2t} & e^{-t} \\ -2e^{t} & 4e^{2t} \end{pmatrix}.$$

Problem 72. Verify that the matrix function

$$X(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

satisfies the differential equation

$$X' = AX, \quad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}.$$

Solution 72. First compute $X'(t)$:

$$X'(t) = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}.$$

Now compute $AX(t)$:

$$AX(t) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}.$$

Multiplying gives

$$AX(t) = \begin{pmatrix} e^{-3t} - 4e^{-3t} & e^{2t} + e^{2t} \\ 4e^{-3t} + 8e^{-3t} & 4e^{2t} - 2e^{2t} \end{pmatrix}.$$

Thus

$$AX(t) = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}.$$

Therefore,

$$X'(t) = AX(t).$$

Problem 73. Let

$$A(t) = \begin{pmatrix} \text{sin}t & -\text{cos}t \\ -\text{cos}t & -\text{sin}t \end{pmatrix}.$$

Does

$$\frac{d}{dt}(A^2)$$

equal $2A\frac{dA}{dt}$, or $2\frac{dA}{dt}A$, or $A\frac{dA}{dt} + \frac{dA}{dt}A$, or none of these? Show work supporting your answer.

Solution 73. Using the product rule for matrix functions,

$$\frac{d}{dt}(A^2) = \frac{dA}{dt}A + A\frac{dA}{dt}.$$

Here

$$A(t) = \begin{pmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{pmatrix},$$

so

$$\frac{dA}{dt} = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}.$$

Now compute

$$A\frac{dA}{dt} = \begin{pmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Also,

$$\frac{dA}{dt}A = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}\begin{pmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore,

$$A\frac{dA}{dt} + \frac{dA}{dt}A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So

$$\frac{d}{dt}(A^2) = A\frac{dA}{dt} + \frac{dA}{dt}A.$$

In this example, this equals the zero matrix. It does not equal $2A\frac{dA}{dt}$ or $2\frac{dA}{dt}A$, since

$$2A\frac{dA}{dt} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \qquad 2\frac{dA}{dt}A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

Problem 74. (a) Transform the fourth-order differential equation

$$y^{(4)} - y'' - y = \sin(t^2)$$

into an equivalent system of first-order differential equations.

(b) Express this first-order system in vector-matrix notation.

Do not attempt to solve either the system or the original differential equation.

Solution 74. Let

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = y'', \qquad x_4 = y^{(3)}.$$

Then

$$x_1' = x_2, \qquad x_2' = x_3, \qquad x_3' = x_4.$$

From the differential equation,

$$y^{(4)} = y'' + y + \sin(t^2).$$

Therefore,

$$x_4' = x_3 + x_1 + \sin(t^2).$$

Thus the equivalent first-order system is

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ x_3' = x_4, \\ x_4' = x_1 + x_3 + \sin(t^2). \end{cases}$$

In vector-matrix notation,

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(t^2) \end{bmatrix}.$$

Problem 75. Are the vectors

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Linearly independent? Show work to justify your answer

Solution 75. Form the matrix using the vectors as columns:

$$W = \det\begin{pmatrix} 0 & 1 & 1 \\ 1 & -4 & 2 \\ 2 & 1 & 0 \end{pmatrix}.$$

Expanding along the first row,

$$\begin{aligned} W &= 0\begin{vmatrix} -4 & 2 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} \\ &= 0 - (0 - 4) + (1 - (-8)) \\ &= 4 + 9 \\ &= 13. \end{aligned}$$

Since

$$W = 13 \neq 0,$$

the vectors are linearly independent.

Problem 76. a) Transform the following differential equation into a system of first order differential equations:

$$y^{(4)} + 3y^{(3)} - y' + y = t^{-1}sin(t)$$

b) Rewrite the system in matrix/vector form

Solution 76. Let

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = y'', \qquad x_4 = y^{(3)}.$$

Then,

$$x_1' = x_2, \qquad x_2' = x_3, \qquad x_3' = x_4, \qquad x_4' = y^{(4)}$$

From the differential equation,

$$y^{(4)} + 3y^{(3)} - y' + y = t^{-1}\sin(t),$$

we solve for $y^{(4)}$:

$$y^{(4)} = -3y^{(3)} + y' - y + t^{-1}\sin(t).$$

Therefore,

$$x_4' = -3x_4 + x_2 - x_1 + t^{-1}\sin(t).$$

Thus, the first order system is

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ x_3' &= x_4, \\ x_4' &= -x_1 + x_2 - 3x_4 + t^{-1}\sin(t). \end{aligned}$$

Now write the system in matrix/vector form:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^{-1}\sin(t) \end{bmatrix}.$$

Problem 77. Given that $x_1(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $x_2(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$ are two linearly independent solutions of

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x$$

Find the fundamental matrix, X, such that $X(0) = I$

Solution 77. Note that there was a typo in the question, the second solution should be

$$x_2(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}.$$

Form a fundamental matrix using the two independent solutions as columns:

$$\Phi(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}.$$

We want a fundamental matrix $X(t)$ such that

$$X(0) = I.$$

There is a formula we can use to calculate this. We first need to find $\Phi(0)$:

$$\Phi(0) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}.$$

Then

$$X(t) = \Phi(t)\Phi(0)^{-1}.$$

Now,

$$\Phi(0)^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} X(t) &= \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix}. \end{aligned}$$

Problem 78. Let $x_1 = y,\ x_2 = y',\ x_3 = y'',\ x_4 = y'''$. Then solve the following differential equation by transforming it into a system of first-order differential equations:

$$y^{(4)} + 2y'' + 6y = t^3.$$

Solution 78. Let

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = y'', \qquad x_4 = y'''.$$

Then,

$$x_1' = x_2, \qquad x_2' = x_3, \qquad x_3' = x_4, \qquad x_4 = y^{(4)}$$

From the differential equation,

$$y^{(4)} + 2y'' + 6y = t^3,$$

solve for $y^{(4)}$:

$$y^{(4)} = -2y'' - 6y + t^3.$$

Therefore,

$$x_4' = -6x_1 - 2x_3 + t^3.$$

Thus, the first order system is

$$
\begin{aligned}
x_1' &= x_2, \\
x_2' &= x_3, \\
x_3' &= x_4, \\
x_4' &= -6x_1 - 2x_3 + t^3.
\end{aligned}
$$

In matrix/vector form,

$$
\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^3 \end{bmatrix}.
$$

Problem 79. Find constants α and β so that

$$\mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

is a solution of

$$\mathbf{x}'(t) = \begin{pmatrix} \alpha & -2 \\ 2 & \beta \end{pmatrix} \mathbf{x}(t).$$

Solution 79. To check if this is a solution, we substitute it into the equation. First,

$$\mathbf{x}'(t) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}.$$

Now compute the right hand side:

$$\begin{pmatrix} \alpha & -2 \\ 2 & \beta \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} 2\alpha - 2 \\ 4 + \beta \end{pmatrix} e^{2t}.$$

Therefore,

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 2\alpha - 2 \\ 4 + \beta \end{pmatrix} e^{2t}.$$

This gives the system

$$
\begin{aligned}
4 &= 2\alpha - 2, \\
2 &= 4 + \beta.
\end{aligned}
$$

Solving,

$$\alpha = 3, \qquad \beta = -2.$$

Problem 80. Determine whether the following vectors are linearly dependent or independent. If they are linearly dependent, find a linear relation among them.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Solution 80. To determine whether the vectors are linearly independent, solve

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}.$$

This gives

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, the corresponding system is

$$\begin{aligned} c_1 + c_3 &= 0, \\ c_1 - c_2 &= 0, \\ c_2 + c_3 &= 0. \end{aligned}$$

Write the coefficient matrix and row reduce:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Row reducing,

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \overset{R2=R2-R1}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\overset{R3=R2+R3}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there is not a pivot in every column, there is a free variable, so the vectors are linearly dependent.

From the reduced system,

$$\begin{aligned} c_1 + c_3 &= 0, \\ c_2 + c_3 &= 0. \end{aligned}$$

Let $c_3 = 1$. Then

$$c_1 = -1, \qquad c_2 = -1.$$

Therefore, a linear relation is

$$-\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}.$$

Equivalently,

$$\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2.$$

Problem 81.

1. Transform the following differential equation

$$y^{(4)} - e^t y''' + 2y = t\cos(t)$$

   into a system of first order differential equations.

2. Write the system that you obtain in part (a) in matrix-vector form.

Solution 81. **Part (a):** To transform a 4th-order differential equation into a system of first-order equations, we define new variables for y and its first three derivatives:

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad x_4 = y'''$$

Now, we find the derivative of each variable:

1. $x_1' = y' = x_2$
2. $x_2' = y'' = x_3$
3. $x_3' = y''' = x_4$
4. $x_4' = y^{(4)}$

From the original equation, solve for $y^{(4)}$:

$$y^{(4)} = e^t y''' - 2y + t\cos(t)$$

Substitute the defined variables into this expression for x_4':

$$x_4' = e^t x_4 - 2x_1 + t\cos(t)$$

The system of first-order equations is:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = -2x_1 + e^t x_4 + t\cos(t) \end{cases}$$

Part (b): We can write this system in the matrix-vector form $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t\cos(t) \end{bmatrix}$$

Problem 82.

1. Express the differential equations

$$x'' + 3x' - y' + 2y = 0, \qquad y'' + x' + 3y' + y = 0$$

as a first order system in normal form. Then write the system in the form

$$\mathbf{x}' = A\mathbf{x}.$$

2. Determine whether the following vector functions are linearly dependent or linearly independent. Justify your answer.

$$\mathbf{x}_1 = \begin{pmatrix} e^t \\ 0 \\ 2e^t \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} -e^t \\ e^t \\ 0 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} e^t \\ e^t \\ 4e^t \end{pmatrix}.$$

Solution 82. **Part (a):** To convert the second-order system into a first-order system, we define the following variables:

$$x_1 = x, \quad x_2 = x', \quad x_3 = y, \quad x_4 = y'$$

Now we find the derivatives:

1. $x_1' = x' = x_2$
2. $x_2' = x'' = -3x' + y' - 2y = -3x_2 + x_4 - 2x_3$
3. $x_3' = y' = x_4$
4. $x_4' = y'' = -x' - 3y' - y = -x_2 - 3x_4 - x_3$

Organizing these into the form $\mathbf{x}' = A\mathbf{x}$:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Part (b): To determine if the vector functions are linearly independent, we calculate the Wronskian $W(t)$, which is the determinant of the matrix formed by these vectors:

$$W(t) = \det \begin{bmatrix} e^t & -e^t & e^t \\ 0 & e^t & e^t \\ 2e^t & 0 & 4e^t \end{bmatrix}$$

Factor out e^t from each column (or the whole matrix):

$$W(t) = (e^t)^3 \det \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

Calculate the determinant using cofactor expansion along the first column:

$$\det = 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} - 0 + 2 \cdot \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$\det = 1(4-0) + 2(-1-1) = 4 + 2(-2) = 4 - 4 = 0$$

Since the Wronskian $W(t) = 0 \cdot e^{3t} = 0$ for all t, the vector functions are **linearly dependent**.

(Note: We can also see that $2\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_3$.)

Problem 83. Rewrite the scalar higher order linear differential equation

$$y''' + 2y'' + 4y = 11t$$

in matrix-vector form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Solution 83. To convert this third-order equation into a system of three first-order equations, we introduce new variables for the function and its derivatives. Let:

$$x_1 = y, \quad x_2 = y', \quad x_3 = y''$$

Now, we find the first derivative of each of these variables:

1. The derivative of x_1 is y', which is our variable x_2:

$$x_1' = x_2$$

2. The derivative of x_2 is y'', which is our variable x_3:

$$x_2' = x_3$$

3. The derivative of x_3 is y'''. From the original differential equation, we solve for y''':

$$y''' = -4y - 2y'' + 11t$$

   Substituting our variables $x_1 = y$ and $x_3 = y''$ into this expression gives:

$$x_3' = -4x_1 - 2x_3 + 11t$$

We can now write these three first-order equations as a system:

$$\begin{cases} x_1' = 0x_1 + 1x_2 + 0x_3 + 0 \\ x_2' = 0x_1 + 0x_2 + 1x_3 + 0 \\ x_3' = -4x_1 + 0x_2 - 2x_3 + 11t \end{cases}$$

In matrix-vector form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, this system is expressed as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 11t \end{bmatrix}$$

Problem 84. Express the system of differential equations

$$x' = 2x + 3y + z$$
$$y' = 4z - 5y$$
$$z' = 3y - 2x$$

in the form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and determine the values for the entries of matrix A.

Solution 84. To find the entries of matrix A, we rewrite each differential equation so that the variables x, y, and z appear in order. For any missing variable, we use a coefficient of 0.

The rewritten system is:

$$x' = (2)x + (3)y + (1)z$$
$$y' = (0)x + (-5)y + (4)z$$
$$z' = (-2)x + (3)y + (0)z$$

By comparing this to the matrix form $A\mathbf{u}$, we can extract the coefficients for each row:

- Row 1 (a, b, c): $a = 2, b = 3, c = 1$
- Row 2 (d, e, f): $d = 0, e = -5, f = 4$
- Row 3 (g, h, i): $g = -2, h = 3, i = 0$

The matrix A is:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -5 & 4 \\ -2 & 3 & 0 \end{bmatrix}$$

The values are: $a = 2, b = 3, c = 1, d = 0, e = -5, f = 4, g = -2, h = 3, i = 0$.

Problem 85. Express the following system of differential equations

$$\begin{cases} \dot{x}_1 = 3x_1 + 2x_2 \\ \dot{x}_2 = 4x_1 + 5x_2 \end{cases}$$

in the matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and determine the values of $a, b, c,$ and d.

Solution 85. To write a system of linear differential equations in matrix form, we extract the coefficients of the variables x_1 and x_2 from each equation.

From the first equation, $\dot{x}_1 = 3x_1 + 2x_2$, the coefficients for the first row of the matrix are:

$$a = 3, \quad b = 2$$

From the second equation, $\dot{x}_2 = 4x_1 + 5x_2$, the coefficients for the second row of the matrix are:

$$c = 4, \quad d = 5$$

Substituting these into the matrix equation gives:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The values are $a = 3, b = 2, c = 4, d = 5$.

Problem 86. Let $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find a matrix A such that the system of differential equations

$$\begin{cases} x' = y + 2x + z \\ y' = 2z - x \\ z' = 4x \end{cases}$$

is equivalent to $\mathbf{u}' = A\mathbf{u}$.

Solution 86. To find the matrix A, we rewrite each equation so that the variables $x, y,$ and z are in the same order. This allows us to easily extract the coefficients for each row of the matrix.

The system can be rewritten as:

$$\begin{aligned} x' &= 2x + 1y + 1z \\ y' &= -1x + 0y + 2z \\ z' &= 4x + 0y + 0z \end{aligned}$$

The matrix A is formed by the coefficients of $x, y,$ and z from each row:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 4 & 0 & 0 \end{bmatrix}$$

Thus, the system in matrix form is:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Problem 87. Express the differential equation

$$y'''(t) + 3y'(t) - 20y(t) = 0$$

as a system of first order differential equations. (Do NOT write the system in matrix form.)

Solution 87. To convert a third-order equation into a system of first-order equations, we introduce new variables for the function and its derivatives. Let:

$$x_1 = y, \quad x_2 = y', \quad x_3 = y''$$

Now, we find the first derivative of each of these new variables:

1. The derivative of x_1 is y', which is our variable x_2:

$$x_1' = x_2$$

2. The derivative of x_2 is y'', which is our variable x_3:

$$x_2' = x_3$$

3. The derivative of x_3 is y'''. From the original differential equation, we solve for y''':

$$y''' = -3y' + 20y$$

   Substituting our variables $x_1 = y$ and $x_2 = y'$ into this expression gives:

$$x_3' = -3x_2 + 20x_1$$

The final system of first-order differential equations is:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 20x_1 - 3x_2 \end{cases}$$

Problem 88. Express the system of differential equations

$$\begin{aligned} x' &= 4x + y + 2z \\ y' &= -3z - 7y \\ z' &= 5y - 6x \end{aligned}$$

in matrix form by finding the matrix A so that:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Solution 88. To find the matrix A, we rewrite the system by explicitly showing the coefficients for all three variables (x, y, z) in every equation. If a variable is missing, its coefficient is 0.

We rewrite the system as:

$$\begin{aligned} x' &= (4)x + (1)y + (2)z \\ y' &= (0)x + (-7)y + (-3)z \\ z' &= (-6)x + (5)y + (0)z \end{aligned}$$

By extracting the coefficients from each row, we construct the matrix A:

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 0 & -7 & -3 \\ -6 & 5 & 0 \end{bmatrix}$$

Thus, the system in matrix form is:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 \\ 0 & -7 & -3 \\ -6 & 5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Problem 89. Express the system of differential equations

$$\begin{aligned} x_1' &= x_1 + 2x_3 \\ x_2' &= 9x_3 - 6x_2 + 3x_1 \\ x_3' &= -4x_2 + 5x_3 \end{aligned}$$

in the form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and determine the values for the entries of matrix A.

Solution 89. To find the entries of matrix A, we must align the variables x_1, x_2, and x_3 for each equation. If a variable is missing, its coefficient is 0.

1. For the first equation, $x_1' = x_1 + 2x_3$:

$$x_1' = (1)x_1 + (0)x_2 + (2)x_3 \Longrightarrow a = 1, b = 0, c = 2$$

2. For the second equation, $x_2' = 9x_3 - 6x_2 + 3x_1$, we reorder the terms:

$$x_2' = (3)x_1 + (-6)x_2 + (9)x_3 \Longrightarrow d = 3, e = -6, f = 9$$

3. For the third equation, $x_3' = -4x_2 + 5x_3$:

$$x_3' = (0)x_1 + (-4)x_2 + (5)x_3 \Longrightarrow g = 0, h = -4, i = 5$$

Substituting these values into the matrix A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -6 & 9 \\ 0 & -4 & 5 \end{bmatrix}$$

The specific values are: $a = 1, b = 0, c = 2, d = 3, e = -6, f = 9, g = 0, h = -4, i = 5$.

Problem 90. Express the system of differential equations

$$x' = x + y + z$$
$$y' = 2z - x$$
$$z' = 4y$$

in matrix form by finding the matrix A so that:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Solution 90. To express the system in matrix form, we first rewrite each equation so that the variables $x, y,$ and z are aligned in columns. If a variable does not appear in an equation, its coefficient is 0.

The rewritten system is:

$$x' = (1)x + (1)y + (1)z$$
$$y' = (-1)x + (0)y + (2)z$$
$$z' = (0)x + (4)y + (0)z$$

The matrix A is formed by taking the coefficients of the variables from each row:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix}$$

Therefore, the system in matrix form is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Problem 91. Express the damped mass-spring oscillator equation

$$(*) \quad my'' + by' + ky = 0$$

as a system of first order differential equations. Do NOT write the system in matrix form.

Solution 91. To convert this second-order equation into a system of two first-order equations, we define new variables for the displacement and its first derivative. Let:

$$x_1 = y \quad \text{and} \quad x_2 = y'$$

Now, we find the derivatives of our new variables:

1. The derivative of x_1 is y', which is simply x_2:

$$x_1' = x_2$$

2. The derivative of x_2 is y''. We find y'' by solving the original equation $(*)$ for the second derivative:

$$my'' = -by' - ky \Longrightarrow y'' = -\frac{b}{m}y' - \frac{k}{m}y$$

Substituting our variables $x_1 = y$ and $x_2 = y'$ into this expression gives:

$$x_2' = -\frac{b}{m}x_2 - \frac{k}{m}x_1$$

Therefore, an equivalent system of first order differential equations is:

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{k}{m}x_1 - \frac{b}{m}x_2 \end{cases}$$

9.4 - 9.5 - Homogeneous Systems with Constant Coefficients

Problem 92. Find the general solution of the system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}.$$

Solution 92. First find the eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix}.$$

Thus

$$(1-\lambda)(-2-\lambda) - 4 = 0.$$

Expanding gives

$$\lambda^2 + \lambda - 6 = 0,$$

so

$$(\lambda + 3)(\lambda - 2) = 0.$$

Therefore,

$$\lambda_1 = -3, \qquad \lambda_2 = 2.$$

For $\lambda_1 = -3$,

$$A + 3I = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}.$$

So

$$4v_1 + v_2 = 0.$$

Choosing $v_1 = 1$, we get $v_2 = -4$, so

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

For $\lambda_2 = 2$,

$$A - 2I = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}.$$

So

$$-v_1 + v_2 = 0.$$

Choosing $v_1 = 1$, we get $v_2 = 1$, so

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the general solution is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Problem 93. Find the solution of the system

$$\begin{aligned} \frac{dx}{dt} &= x + 6y, \\ \frac{dy}{dt} &= x - 4y, \end{aligned} \qquad x(0) = 5, \quad y(0) = 9.$$

Solution 93. We can write the system as

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{pmatrix} 1 & 6 \\ 1 & -4 \end{pmatrix}.$$

First find the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 6 \\ 1 & -4 - \lambda \end{vmatrix}.$$

Thus

$$(1 - \lambda)(-4 - \lambda) - 6 = 0.$$

Expanding gives

$$\lambda^2 + 3\lambda - 10 = 0,$$

so

$$(\lambda - 2)(\lambda + 5) = 0.$$

Therefore,

$$\lambda_1 = 2, \qquad \lambda_2 = -5.$$

For $\lambda_1 = 2$,

$$A - 2I = \begin{pmatrix} -1 & 6 \\ 1 & -6 \end{pmatrix}.$$

Thus

$$-v_1 + 6v_2 = 0.$$

Choosing $v_2 = 1$, we get $v_1 = 6$, so

$$\mathbf{v}_1 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -5$,

$$A + 5I = \begin{pmatrix} 6 & 6 \\ 1 & 1 \end{pmatrix}.$$

Thus

$$6v_1 + 6v_2 = 0,$$

so

$$v_1 = -v_2.$$

Choosing $v_2 = 1$, we get $v_1 = -1$, so

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore,

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 6 \\ 1 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Using the initial conditions $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$, we get

$$\begin{pmatrix} 5 \\ 9 \end{pmatrix} = c_1 \begin{pmatrix} 6 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus

$$6c_1 - c_2 = 5, \qquad c_1 + c_2 = 9.$$

Solving gives

$$c_1 = 2, \qquad c_2 = 7.$$

Therefore,

$$\mathbf{x}(t) = 2e^{2t}\begin{pmatrix}6\\1\end{pmatrix} + 7e^{-5t}\begin{pmatrix}-1\\1\end{pmatrix}.$$

Equivalently,

$$\begin{cases} x(t) = 12e^{2t} - 7e^{-5t}, \\ y(t) = 2e^{2t} + 7e^{-5t}. \end{cases}$$

Problem 94. Solve the initial value problem

$$\begin{aligned} x' &= x + y, \\ y' &= 4x + y, \end{aligned} \quad x(0) = 2, \quad y(0) = 0.$$

Solution 94. We can write the system as

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{pmatrix}1 & 1\\4 & 1\end{pmatrix}.$$

First find the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix}1-\lambda & 1\\4 & 1-\lambda\end{vmatrix}.$$

Thus

$$(1-\lambda)^2 - 4 = 0.$$

So

$$1 - \lambda = \pm 2.$$

Therefore,

$$\lambda_1 = 3, \qquad \lambda_2 = -1.$$

For $\lambda_1 = 3$,

$$A - 3I = \begin{pmatrix}-2 & 1\\4 & -2\end{pmatrix}.$$

Thus

$$-2v_1 + v_2 = 0.$$

Choosing $v_1 = 1$, we get $v_2 = 2$, so

$$\mathbf{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}.$$

For $\lambda_2 = -1$,

$$A + I = \begin{pmatrix}2 & 1\\4 & 2\end{pmatrix}.$$

Thus

$$2v_1 + v_2 = 0.$$

Choosing $v_1 = 1$, we get $v_2 = -2$, so

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Therefore,

$$\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Using the initial conditions $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, we get

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Thus

$$c_1 + c_2 = 2, \qquad 2c_1 - 2c_2 = 0.$$

Solving gives

$$c_1 = 1, \qquad c_2 = 1.$$

Therefore,

$$\mathbf{x}(t) = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Equivalently,

$$\begin{cases} x(t) = e^{3t} + e^{-t}, \\ y(t) = 2e^{3t} - 2e^{-t}. \end{cases}$$

Problem 95. Find the general solution of the system

$$x' = 3x + 6y,$$
$$y' = -x - 2y.$$

Solution 95. We can write the system as

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}.$$

First find the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 6 \\ -1 & -2 - \lambda \end{vmatrix}.$$

Thus

$$(3 - \lambda)(-2 - \lambda) + 6 = 0.$$

Expanding gives

$$\lambda^2 - \lambda = 0,$$

so

$$\lambda(\lambda - 1) = 0.$$

Therefore,

$$\lambda_1 = 0, \qquad \lambda_2 = 1.$$

For $\lambda_1 = 0$,

$$A = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}.$$

Thus

$$3v_1 + 6v_2 = 0,$$

so

$$v_1 = -2v_2.$$

Choosing $v_2 = 1$, we get

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1$,

$$A - I = \begin{pmatrix} 2 & 6 \\ -1 & -3 \end{pmatrix}.$$

Thus

$$2v_1 + 6v_2 = 0,$$

so

$$v_1 = -3v_2.$$

Choosing $v_2 = 1$, we get

$$\mathbf{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Therefore, the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Equivalently,

$$\begin{cases} x(t) = -2c_1 - 3c_2e^t, \\ y(t) = c_1 + c_2e^t. \end{cases}$$

Problem 96. Find the general solution of

$$x' = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} x$$

Solution 96. To find the general solution, we first find the eigenvalues of the matrix $A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$ by solving the characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -\lambda & -4 \\ 1 & -\lambda \end{bmatrix} = (-\lambda)(-\lambda) - (-4)(1) = \lambda^2 + 4 = 0$$

Solving for λ, we get purely imaginary eigenvalues:

$$\lambda^2 = -4 \Longrightarrow \lambda = \pm 2i$$

Next, we find the eigenvector $\mathbf{v}$ corresponding to $\lambda = 2i$ by solving $(A - 2iI)\mathbf{v} = \mathbf{0}$:

$$\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the second row, we have $v_1 - 2iv_2 = 0$, which gives $v_1 = 2iv_2$. Choosing $v_2 = 1$, we get the eigenvector:

$$\mathbf{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

A complex-valued solution is $e^{2it}\mathbf{v}$. Using Euler's formula $e^{2it} = \cos(2t) + i\sin(2t)$:

$$\mathbf{x}(t) = (\cos(2t) + i\sin(2t)) \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \begin{bmatrix} 2i\cos(2t) - 2\sin(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

Separate the real and imaginary parts to find two linearly independent real solutions:

$$\text{Re}\{\mathbf{x}(t)\} = \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix}, \quad \text{Im}\{\mathbf{x}(t)\} = \begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix}$$

The general solution is a linear combination of these two parts:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 \begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix}$$

Problem 97. Solve the initial value problem

$$\begin{aligned} x' &= -5x + y, \quad x(0) = 2, \\ y' &= -3x - y, \quad y(0) = -1. \end{aligned}$$

Describe the behavior of the solution as $t \to \infty$.

Solution 97. We can write the system as

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{pmatrix} -5 & 1 \\ -3 & -1 \end{pmatrix}.$$

First find the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 1 \\ -3 & -1-\lambda \end{vmatrix}.$$

Thus

$$(-5-\lambda)(-1-\lambda) + 3 = 0.$$

Expanding gives

$$\lambda^2 + 6\lambda + 8 = 0,$$

so

$$(\lambda+2)(\lambda+4) = 0.$$

Therefore,

$$\lambda_1 = -2, \qquad \lambda_2 = -4.$$

For $\lambda_1 = -2$,

$$A + 2I = \begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix}.$$

Thus

$$-3v_1 + v_2 = 0.$$

Choosing $v_1 = 1$, we get

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

For $\lambda_2 = -4$,

$$A + 4I = \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix}.$$

Thus

$$-v_1 + v_2 = 0.$$

Choosing $v_1 = 1$, we get

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore,

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Using the initial conditions $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, we get

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus

$$c_1 + c_2 = 2, \qquad 3c_1 + c_2 = -1.$$

Solving gives

$$c_1 = -\frac{3}{2}, \qquad c_2 = \frac{7}{2}.$$

Therefore,

$$\mathbf{x}(t) = -\frac{3}{2} e^{-2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{7}{2} e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Equivalently,

$$\begin{cases} x(t) = -\frac{3}{2} e^{-2t} + \frac{7}{2} e^{-4t}, \\ y(t) = -\frac{9}{2} e^{-2t} + \frac{7}{2} e^{-4t}. \end{cases}$$

Since both eigenvalues are negative,

$$\mathbf{x}(t) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{as } t \to \infty.$$

Thus the solution approaches the origin as $t \to \infty$.

Problem 98. (a) Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) Describe the behavior of the solution as $t \to \infty$.

Solution 98. First find the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & -3 \\ 8 & -6 - \lambda \end{vmatrix}.$$

Thus

$$(4 - \lambda)(-6 - \lambda) + 24 = 0.$$

Expanding gives

$$\lambda^2 + 2\lambda = 0,$$

so

$$\lambda(\lambda + 2) = 0.$$

Therefore,

$$\lambda_1 = 0, \qquad \lambda_2 = -2.$$

For $\lambda_1 = 0$,

$$A = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}.$$

Thus

$$4v_1 - 3v_2 = 0.$$

Choosing $v_1 = 3$, we get $v_2 = 4$, so

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

For $\lambda_2 = -2$,

$$A + 2I = \begin{pmatrix} 6 & -3 \\ 8 & -4 \end{pmatrix}.$$

Thus

$$2v_1 - v_2 = 0.$$

Choosing $v_1 = 1$, we get $v_2 = 2$, so

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore,

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Using $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we get

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus

$$3c_1 + c_2 = 1, \qquad 4c_1 + 2c_2 = 0.$$

Solving gives

$$c_1 = 1, \qquad c_2 = -2.$$

Therefore,

$$\mathbf{x}(t) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - 2e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Equivalently,

$$\begin{cases} x_1(t) = 3 - 2e^{-2t}, \\ x_2(t) = 4 - 4e^{-2t}. \end{cases}$$

Since $e^{-2t} \to 0$ as $t \to \infty$,

$$\mathbf{x}(t) \to \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Thus the solution approaches the equilibrium point $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ as $t \to \infty$.

Problem 99. Find two linearly independent vector solutions of

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix},$$

and calculate their Wronskian.

Solution 99. First find the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -1 \\ 4 & -2 - \lambda \end{vmatrix}.$$

Thus

$$(3 - \lambda)(-2 - \lambda) + 4 = 0.$$

Expanding gives

$$\lambda^2 - \lambda - 2 = 0,$$

so

$$(\lambda - 2)(\lambda + 1) = 0.$$

Therefore,

$$\lambda_1 = 2, \qquad \lambda_2 = -1.$$

For $\lambda_1 = 2$,

$$A - 2I = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}.$$

Thus

$$v_1 - v_2 = 0.$$

Choosing $v_1 = 1$, we get $v_2 = 1$, so

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -1$,

$$A + I = \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}.$$

Thus

$$4v_1 - v_2 = 0.$$

Choosing $v_1 = 1$, we get $v_2 = 4$, so

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Therefore, two linearly independent vector solutions are

$$\mathbf{x}_1(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

The Wronskian is

$$W(t) = \det \begin{pmatrix} e^{2t} & e^{-t} \\ e^{2t} & 4e^{-t} \end{pmatrix}.$$

Thus

$$W(t) = 4e^t - e^t = 3e^t.$$

Problem 100. Given that $\Phi(t) = \begin{bmatrix} e^{-t} & e^t \\ 3e^{-t} & e^t \end{bmatrix}$ is a fundamental matrix for

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

Solve the initial value problem

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Solution 100. The general solution is in the form: $x = \Phi(t)c$, so:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-t} & e^t \\ 3e^{-t} & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Sub in the initial conditions, $x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Then we obtain:

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Which leads to:

$$2 = c_1 + c_2$$
$$-1 = 3c_1 + c_2$$

Which results in $c_1 = -\frac{3}{2}$ and $c_2 = \frac{7}{2}$. Thus, the general solution is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{t} \\ 3e^{-t} & e^{t} \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ \frac{7}{2} \end{bmatrix}$$

Problem 101. Solve the initial value problem

$$x'(t) = -2x - y$$
$$y'(t) = -7x + 4y$$

with $x(0) = 1, y(0) = 3$

Solution 101. Write the system in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -2-\lambda & -1 \\ -7 & 4-\lambda \end{vmatrix} \\ &= (-2-\lambda)(4-\lambda) - 7 \\ &= \lambda^2 - 2\lambda - 15 \\ &= (\lambda - 5)(\lambda + 3). \end{aligned}$$

Thus,

$$\lambda_1 = 5, \qquad \lambda_2 = -3.$$

For $\lambda_1 = 5$,

$$\begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which leads to the equation $-7v_1 - v_2 = 0$, which gives an eigenvector:

$$V_1 = \begin{bmatrix} 1 \\ -7 \end{bmatrix}.$$

For $\lambda_2 = -3$,

$$\begin{bmatrix} 1 & -1 \\ -7 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which leads to the equation $v_1 - v_2 = 0$, which gives an eigenvector:

$$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{5t} \begin{bmatrix} 1 \\ -7 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Sub in the initial condition

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -7 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This gives

$$\begin{aligned} 1 &= c_1 + c_2, \\ 3 &= -7c_1 + c_2. \end{aligned}$$

Solving,

$$c_1 = -\frac{1}{4}, \qquad c_2 = \frac{5}{4}.$$

Thus, the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{4} e^{5t} \begin{bmatrix} 1 \\ -7 \end{bmatrix} + \frac{5}{4} e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Problem 102. Find the general solution of the system of differential equations

$$x' = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} x$$

Solution 102. Write the system as

$$x' = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} x.$$

Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} \\ &= (3-\lambda)(4-\lambda) - 2 \\ &= \lambda^2 - 7\lambda + 10 \\ &= (\lambda - 5)(\lambda - 2). \end{aligned}$$

Thus,

$$\lambda_1 = 5, \qquad \lambda_2 = 2.$$

For $\lambda_1 = 5$,

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This leads to the equation

$$-2v_1 + 2v_2 = 0,$$

so $v_1 = v_2$. Therefore, an eigenvector is

$$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 2$,

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This leads to the equation

$$v_1 + 2v_2 = 0.$$

Taking $v_2 = 1$, we get $v_1 = -2$. Therefore, an eigenvector is

$$V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Therefore, the general solution is

$$x(t) = c_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Problem 103. Find the general solution of the system of differential equations

$$x' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} x$$

Solution 103. Write the system as

$$x' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} x.$$

Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{vmatrix} \\ &= (3 - \lambda)(-1 - \lambda) + 8 \\ &= \lambda^2 - 2\lambda + 5. \end{aligned}$$

Thus,

$$\lambda = 1 \pm 2i.$$

For $\lambda_1 = 1 + 2i$,

$$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This leads to the equation

$$(2-2i)v_1 - 2v_2 = 0.$$

Taking $v_1 = 1$, we get

$$v_2 = 1 - i.$$

Therefore, an eigenvector is

$$V_1 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}.$$

Now write

$$e^{(1+2i)t}\begin{bmatrix} 1 \\ 1-i \end{bmatrix} = e^t(\cos(2t) + i\sin(2t))\begin{bmatrix} 1 \\ 1-i \end{bmatrix}.$$

Multiplying out,

$$e^{(1+2i)t}\begin{bmatrix} 1 \\ 1-i \end{bmatrix} = e^t\begin{bmatrix} \cos(2t) + i\sin(2t) \\ \cos(2t) + \sin(2t) + i(\sin(2t) - \cos(2t)) \end{bmatrix}.$$

Therefore, the two real-valued solutions are the real and imaginary parts:

$$x^{(1)}(t) = e^t\begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix},$$

and

$$x^{(2)}(t) = e^t\begin{bmatrix} \sin(2t) \\ \sin(2t) - \cos(2t) \end{bmatrix}.$$

Thus, the general solution is

$$x(t) = c_1 e^t\begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} + c_2 e^t\begin{bmatrix} \sin(2t) \\ \sin(2t) - \cos(2t) \end{bmatrix}.$$

Problem 104. Find the general solution of the system

$$\frac{dx_1}{dt} = -2x_2 + x_1, \qquad \frac{dx_2}{dt} = 3x_1 - 4x_2.$$

Solution 104. Write the system as

$$x' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} x.$$

Find the eigenvalues:

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix}1-\lambda & -2\\ 3 & -4-\lambda\end{vmatrix}\\ &= (1-\lambda)(-4-\lambda)+6\\ &= \lambda^2+3\lambda+2\\ &= (\lambda+1)(\lambda+2).\end{aligned}$$

Thus,

$$\lambda_1=-1,\qquad \lambda_2=-2.$$

For $\lambda_1=-1$,

$$\begin{bmatrix}2 & -2\\ 3 & -3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

This leads to the equation

$$2v_1-2v_2=0,$$

so $v_1=v_2$. Therefore, an eigenvector is

$$V_1=\begin{bmatrix}1\\ 1\end{bmatrix}.$$

For $\lambda_2=-2$,

$$\begin{bmatrix}3 & -2\\ 3 & -2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

This leads to the equation

$$3v_1-2v_2=0.$$

Taking $v_1=2$, we get $v_2=3$. Therefore, an eigenvector is

$$V_2=\begin{bmatrix}2\\ 3\end{bmatrix}.$$

Thus, the general solution is

$$x(t)=c_1e^{-t}\begin{bmatrix}1\\ 1\end{bmatrix}+c_2e^{-2t}\begin{bmatrix}2\\ 3\end{bmatrix}.$$

Problem 105. Solve the initial value problem

$$\mathbf{x}'=\begin{pmatrix}1 & -2\\ 1 & -1\end{pmatrix}\mathbf{x},\qquad \mathbf{x}(0)=\begin{pmatrix}3\\ 1\end{pmatrix}.$$

Solution 105. Write the system as

$$\mathbf{x}'=\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}\mathbf{x}.$$

Find the eigenvalues:

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda)+2 \\ &= \lambda^2+1.\end{aligned}$$

Thus,

$$\lambda = \pm i.$$

For $\lambda_1 = i$,

$$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This leads to the equation

$$v_1 - (1+i)v_2 = 0.$$

Taking $v_2 = 1$, we get

$$v_1 = 1+i.$$

Therefore, an eigenvector is

$$V_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}.$$

Now write

$$e^{it}\begin{bmatrix} 1+i \\ 1 \end{bmatrix} = (\cos(t) + i\sin(t))\begin{bmatrix} 1+i \\ 1 \end{bmatrix}.$$

Multiplying out,

$$e^{it}\begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(t) - \sin(t) + i(\sin(t) + \cos(t)) \\ \cos(t) + i\sin(t) \end{bmatrix}.$$

Therefore, the two real-valued solutions are the real and imaginary parts:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix},$$

and

$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} \sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}.$$

Therefore, the general solution is

$$\mathbf{x}(t) = c_1\begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2\begin{bmatrix} \sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}.$$

Sub in the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Since

$$\mathbf{x}^{(1)}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}^{(2)}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

we get

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

This gives

$$\begin{aligned} c_1 + c_2 &= 3, \\ c_1 &= 1. \end{aligned}$$

Hence,

$$c_1 = 1, \qquad c_2 = 2.$$

Thus, the solution is

$$\mathbf{x}(t) = \begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + 2 \begin{bmatrix} \sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}.$$

Problem 106. Find the general solution of the following system of differential equations:

$$\begin{cases} x_1'(t) = x_1(t) + 2x_2(t), \\ x_2'(t) = 2x_1(t) + x_2(t). \end{cases}$$

Solution 106. Write the system as

$$x' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x.$$

Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)^2 - 4 \\ &= \lambda^2 - 2\lambda - 3 \\ &= (\lambda - 3)(\lambda + 1). \end{aligned}$$

Thus,

$$\lambda_1 = 3, \qquad \lambda_2 = -1.$$

For $\lambda_1 = 3$,

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This leads to

$$-2v_1 + 2v_2 = 0,$$

so $v_1 = v_2$. Therefore, an eigenvector is

$$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -1$,

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This leads to

$$2v_1 + 2v_2 = 0.$$

Taking $v_2 = 1$, we get $v_1 = -1$. Therefore, an eigenvector is

$$V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Thus, the general solution is

$$x(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Problem 107. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & 2 \\ 10 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution 107. Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 2 \\ 10 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)(1-\lambda) - 20 \\ &= \lambda^2 - 3\lambda - 18 \\ &= (\lambda - 6)(\lambda + 3). \end{aligned}$$

Thus,

$$\lambda_1 = 6, \qquad \lambda_2 = -3.$$

For $\lambda_1 = 6$,

$$\begin{bmatrix} -4 & 2 \\ 10 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By using the row operations $R2 = 5R1 + 2R2$, this leads to

$$-4v_1 + 2v_2 = 0,$$

so $v_2 = 2v_1$. Therefore, an eigenvector is

$$V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For $\lambda_2 = -3$,

$$\begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By using the row operations $R2 = (-2)R1 + R2$, this leads to

$$5v_1 + 2v_2 = 0.$$

Taking $v_1 = 2$, we get $v_2 = -5$. Therefore, an eigenvector is

$$V_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

Therefore, the general solution is

$$\mathbf{x}(t) = c_1 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

Sub in the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

This gives

$$\begin{aligned} 1 &= c_1 + 2c_2, \\ -1 &= 2c_1 - 5c_2. \end{aligned}$$

Solving,

$$c_1 = \frac{3}{9} = \frac{1}{3}, \qquad c_2 = \frac{1}{3}.$$

Thus, the solution is

$$\mathbf{x}(t) = \frac{1}{3} e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

Problem 108. Find the general solution to the differential equation system:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -1 & 1 \\ 8 & 1 \end{pmatrix}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Solution 108. To solve the system, we first find the eigenvalues of matrix A by solving the characteristic equation $\det(A - \lambda I) = 0$:

$$\det\begin{pmatrix} -1-\lambda & 1 \\ 8 & 1-\lambda \end{pmatrix} = (-1-\lambda)(1-\lambda) - 8 = 0$$

Expanding the product:

$$-(1+\lambda)(1-\lambda) - 8 = -(1-\lambda^2) - 8 = \lambda^2 - 1 - 8 = \lambda^2 - 9 = 0$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -3$.

Next, we find the eigenvector $\vec{\xi}$ for each eigenvalue by solving $(A - \lambda I)\vec{\xi} = \mathbf{0}$.

For $\lambda_1 = 3$:

$$\begin{pmatrix} -1-3 & 1 \\ 8 & 1-3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 8 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the first row: $-4\xi_1 + \xi_2 = 0 \Longrightarrow \xi_2 = 4\xi_1$. Setting $\xi_1 = 1$, we get the eigenvector:

$$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

For $\lambda_2 = -3$:

$$\begin{pmatrix} -1-(-3) & 1 \\ 8 & 1-(-3) \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the first row: $2\xi_1 + \xi_2 = 0 \Longrightarrow \xi_2 = -2\xi_1$. Setting $\xi_1 = 1$, we get the eigenvector:

$$\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The general solution is the linear combination of the fundamental solutions $\mathbf{x}^{(1)}(t) = e^{\lambda_1 t}\vec{\xi}^{(1)}$ and $\mathbf{x}^{(2)}(t) = e^{\lambda_2 t}\vec{\xi}^{(2)}$:

$$\mathbf{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Problem 109. Find the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Solution 109. First, we find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$:

$$\det\begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - (-1)^2 = 0$$

Expanding and simplifying:

$$9 - 6\lambda + \lambda^2 - 1 = \lambda^2 - 6\lambda + 8 = 0$$

Factoring the quadratic:

$$(\lambda-2)(\lambda-4)=0$$

The eigenvalues are $\lambda_1=2$ and $\lambda_2=4$.

Next, we find the eigenvectors $\vec{v}$ for each eigenvalue.

For $\lambda_1=2$: We solve $(A-2I)\vec{v}=\mathbf{0}$:

$$\begin{bmatrix}3-2 & -1\\ -1 & 3-2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

This gives the equation $v_1-v_2=0 \Longrightarrow v_1=v_2$. Letting $v_2=1$, we get:

$$\vec{v}^{(1)}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

For $\lambda_2=4$: We solve $(A-4I)\vec{v}=\mathbf{0}$:

$$\begin{bmatrix}3-4 & -1\\ -1 & 3-4\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

This gives the equation $-v_1-v_2=0 \Longrightarrow v_1=-v_2$. Letting $v_2=1$, we get:

$$\vec{v}^{(2)}=\begin{bmatrix}-1\\ 1\end{bmatrix}$$

The general solution is:

$$\mathbf{x}(t)=C_1e^{2t}\begin{bmatrix}1\\ 1\end{bmatrix}+C_2e^{4t}\begin{bmatrix}-1\\ 1\end{bmatrix}$$

Problem 110. Find the general solution of the differential equation system:

$$\vec{x}'=A\vec{x},\quad A=\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}$$

Solution 110. First, we find the eigenvalues of matrix A by solving the characteristic equation $\det(A-\lambda I)=0$:

$$\det\begin{bmatrix}3-\lambda & -1\\ -1 & 3-\lambda\end{bmatrix}=(3-\lambda)^2-(-1)(-1)=0$$

Expanding the binomial and simplifying:

$$\lambda^2-6\lambda+9-1=\lambda^2-6\lambda+8=0$$

Factoring the quadratic equation:

$$(\lambda-4)(\lambda-2)=0$$

The eigenvalues are $\lambda_1=2$ and $\lambda_2=4$.

Next, we find the eigenvectors $\vec{v}$ for each eigenvalue by solving $(A-\lambda I)\vec{v}=\vec{0}$.

For $\lambda_1 = 2$**:**

$$\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first row, we get the equation $a - b = 0$, which implies $a = b$. Let $b = 1$, then:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 4$**:**

$$\begin{bmatrix} 3-4 & -1 \\ -1 & 3-4 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first row, we get the equation $-c - d = 0$, which implies $c = -d$. Let $d = 1$, then:

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The general solution is a linear combination of the fundamental solutions $\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1$ and $\vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2$:

$$\vec{x}(t) = C_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Problem 111. Find the general solution of the differential equation system:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

Solution 111. First, we determine the eigenvalues of matrix A by solving the characteristic equation $\det(A - \lambda I) = 0$:

$$\det\begin{bmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda) - 8 = 0$$

Expanding the expression:

$$3 - 3\lambda - \lambda + \lambda^2 - 8 = \lambda^2 - 4\lambda - 5 = 0$$

Factoring the quadratic:

$$(\lambda - 5)(\lambda + 1) = 0$$

The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -1$.

Next, we find the eigenvectors $\vec{u}$ for each eigenvalue by solving $(A - \lambda I)\vec{u} = \vec{0}$.

For $\lambda_1 = 5$**:**

$$\begin{bmatrix} 3-5 & 2 \\ 4 & 1-5 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first row gives $-2u_1 + 2u_2 = 0$, which simplifies to $u_1 = u_2$. Letting $u_2 = 1$, we get:

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -1$**:**

$$\begin{bmatrix} 3-(-1) & 2 \\ 4 & 1-(-1) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first row gives $4u_1 + 2u_2 = 0$, which simplifies to $2u_1 = -u_2$. Letting $u_1 = -1$, we find $u_2 = 2$:

$$\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The general solution is a linear combination of the fundamental solutions:

$$\mathbf{x}(t) = c_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Problem 112. Find the general solution of the differential equation system:

$$\vec{x}' = A\vec{x}, \quad A = \begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}$$

Solution 112. We assume a solution of the form $\vec{x} = \vec{k}e^{\lambda t}$, which leads to the eigenvalue problem $A\vec{k} = \lambda\vec{k}$. First, we solve the characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -1-\lambda & 1 \\ 8 & 1-\lambda \end{bmatrix} = (-1-\lambda)(1-\lambda) - 8 = \lambda^2 - 1 - 8 = \lambda^2 - 9 = 0$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -3$.

Next, we find the eigenvectors $\vec{k}$ for each eigenvalue by solving $(A - \lambda I)\vec{k} = \vec{0}$.

Eigenvector Calculation for $\lambda_1 = 3$**:**

$$\begin{bmatrix} -4 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow -4k_1 + k_2 = 0 \Longrightarrow k_2 = 4k_1$$

Choosing $k_1 = 1$, we get $\vec{k}^{(1)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Eigenvector Calculation for $\lambda_2 = -3$**:**

$$\begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow 2k_1 + k_2 = 0 \Longrightarrow k_2 = -2k_1$$

Choosing $k_1 = 1$, we get $\vec{k}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

The individual solutions are $\vec{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{3t}$ and $\vec{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-3t}$. We check the Wronskian to ensure they are linearly independent:

$$W(t) = \det \begin{bmatrix} e^{3t} & e^{-3t} \\ 4e^{3t} & -2e^{-3t} \end{bmatrix} = -2e^0 - 4e^0 = -6 \neq 0$$

Since $W(t) \neq 0$, these form a fundamental set of solutions. The general solution is:

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-3t}$$

where c_1 and c_2 are arbitrary constants.

Problem 113. Find the general solution of the differential equation system:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Solution 113. First, we determine the eigenvalues of matrix A by solving the characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - (-1)^2 = 0$$

Expanding and simplifying the equation:

$$(4 - 4\lambda + \lambda^2) - 1 = \lambda^2 - 4\lambda + 3 = 0$$

Factoring the quadratic equation:

$$(\lambda - 3)(\lambda - 1) = 0$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$.

Next, we find the eigenvectors $\vec{v}$ for each eigenvalue by solving $(A - \lambda I)\vec{v} = \vec{0}$.

For $\lambda_1 = 3$:

$$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first row gives $-a - b = 0$, which implies $a = -b$. Letting $b = 1$, we get the eigenvector:

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 1$:

$$\begin{bmatrix} 2-1 & -1 \\ -1 & 2-1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first row gives $c - d = 0$, which implies $c = d$. Letting $d = 1$, we get the eigenvector:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general solution is a linear combination of the fundamental solutions:

$$\mathbf{x}(t) = c_1 e^{3t}\begin{bmatrix}-1\\1\end{bmatrix} + c_2 e^{t}\begin{bmatrix}1\\1\end{bmatrix}$$

Problem 114. Given that $\mathbf{x} = \begin{bmatrix}e^{6t}\\e^{6t}\end{bmatrix}$ is a solution of the following system

$$\mathbf{x}' = \begin{bmatrix}1 & 5\\5 & 1\end{bmatrix}\mathbf{x},$$

find the general solution of system (1).

Solution 114. We seek solutions of the form $\mathbf{x}(t) = \vec{k}e^{\lambda t}$. From the given solution, we can identify one eigenvalue $\lambda_2 = 6$ with corresponding eigenvector $\vec{k}^{(2)} = \begin{bmatrix}1\\1\end{bmatrix}$. To find the general solution, we find the other eigenvalue by solving the characteristic equation $\det(A - \lambda I) = 0$:

$$\det\begin{bmatrix}1-\lambda & 5\\5 & 1-\lambda\end{bmatrix} = (1-\lambda)^2 - 25 = 0$$

$$\lambda^2 - 2\lambda + 1 - 25 = \lambda^2 - 2\lambda - 24 = 0$$

Factoring the quadratic:

$$(\lambda - 6)(\lambda + 4) = 0$$

The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 6$.

Next, we find the eigenvector $\vec{k}^{(1)}$ for $\lambda_1 = -4$ by solving $(A - (-4)I)\vec{k} = \mathbf{0}$:

$$\begin{bmatrix}1+4 & 5\\5 & 1+4\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}5 & 5\\5 & 5\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

The first row gives $5k_1 + 5k_2 = 0 \Longrightarrow k_2 = -k_1$. Setting $k_1 = 1$, we get:

$$\vec{k}^{(1)} = \begin{bmatrix}1\\-1\end{bmatrix}$$

The two fundamental solutions are:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix}1\\-1\end{bmatrix}e^{-4t} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix}1\\1\end{bmatrix}e^{6t}$$

We verify they form a fundamental set by calculating the Wronskian at $t = 0$:

$$W(0) = \det\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} = 1 - (-1) = 2 \neq 0$$

The general solution is the linear combination of these fundamental solutions:

$$\mathbf{x}(t) = c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{-4t} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{6t}$$

Problem 115. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$$

Solution 115. **1. Find the Eigenvalues:** We solve the characteristic equation $\det(A - \lambda I) = 0$:

$$\det\begin{pmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{pmatrix} = (1-\lambda)(5-\lambda) - (-1)(3) = 0$$

Expanding the product:

$$5 - 5\lambda - \lambda + \lambda^2 + 3 = \lambda^2 - 6\lambda + 8 = 0$$

Factoring the quadratic:

$$(\lambda - 2)(\lambda - 4) = 0$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 4$.

2. Find the Eigenvectors: For each eigenvalue, we solve $(A - \lambda I)\vec{v} = \vec{0}$.

For $\lambda_1 = 2$:

$$\begin{pmatrix} 1-2 & 3 \\ -1 & 5-2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Both rows give the equation $-a + 3b = 0$, which means $a = 3b$. Choosing $b = 1$, we get the eigenvector:

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4$:

$$\begin{pmatrix} 1-4 & 3 \\ -1 & 5-4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Both rows give the equation $-a + b = 0$, which means $a = b$. Choosing $b = 1$, we get the eigenvector:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

3. General Solution: Since the eigenvalues are distinct, the corresponding eigenvectors are linearly independent. The general solution is:

$$\mathbf{x}(t) = C_1 e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + C_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where C_1 and C_2 are arbitrary constants.

Problem 116. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

Solution 116. We seek solutions of the form $\mathbf{x} = \vec{k}e^{\lambda t}$. Substituting this into the system $\mathbf{x}' = A\mathbf{x}$ leads to the eigenvalue equation $A\vec{k} = \lambda\vec{k}$. To find the eigenvalues, we solve the characteristic equation $\det(A - \lambda I) = 0$:

$$\det\begin{bmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix} = (2-\lambda)(-2-\lambda) - (-3)(1) = 0$$

$$\lambda^2 - 4 + 3 = \lambda^2 - 1 = 0$$

Factoring the quadratic gives $(\lambda - 1)(\lambda + 1) = 0$, so the eigenvalues are:

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

Next, we find the corresponding eigenvectors $\vec{k}$ by solving $(A - \lambda I)\vec{k} = \mathbf{0}$:

1. For $\lambda_1 = 1$:

$$\begin{bmatrix} 2-1 & -3 \\ 1 & -2-1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   The equation $k_1 - 3k_2 = 0$ implies $k_1 = 3k_2$. Choosing $k_2 = 1$ gives $\vec{k}^{(1)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

2. For $\lambda_2 = -1$:

$$\begin{bmatrix} 2+1 & -3 \\ 1 & -2+1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   The equation $k_1 - k_2 = 0$ implies $k_1 = k_2$. Choosing $k_1 = 1$ gives $\vec{k}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The fundamental solutions are:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}e^t \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t}$$

We verify linear independence by calculating the Wronskian $W(t) = \det[\mathbf{x}^{(1)}(t)\ \mathbf{x}^{(2)}(t)]$:

$$W(t) = \det\begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix} = (3e^t)(e^{-t}) - (e^t)(e^{-t}) = 3 - 1 = 2 \neq 0$$

The solutions are linearly independent on $(-\infty, \infty)$. The general solution is:

$$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^t + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t}$$