Chapter 12

Wiener Processes and Itô's Lemma

176

Definition 12.1

A stochastic process is a family of random variables X=X(t), where t could be integers or real numbers.

177

Remark 12.2 (Normal RV)

$X \sim N(0,1)$
$P(X \le 0)=$
$P(X \le 0.2)=$
$P(X \le 0.22)=$
$P(X \le -0.2)=$
$P(X \le x)=0.95$

$X \sim N(2,4)$
$P(X \le 2.2)=$

$P(X \le 1)=$

$P(X \le x)=0.95$

If $X \sim N(\mu,\sigma)$, then $(X-\mu)/\sigma \sim N(0,1)$

178

Definition 12.3

A stochastic process follows a Wiener process if

- the change ΔW during a small period of time Δt is

$\Delta W=\varepsilon(\Delta t)^{1/2}$, where $\varepsilon \sim N(0,1)$,

- the values of ΔW for any two different short intervals of time Δt are independent.

179

Example 12.4

Suppose W follows a Wiener process and time is measured in years. Suppose the value of W is initially 25. What is the value of W at the end of one year? What is the value of W at the end of five years? Find $P(W(1)>26)$ and $P(W(5)>26)$.

180

Definition 12.5

A stochastic process X follows a generalized Wiener process with drift rate a and variance rate b^2 if

$dX=adt+bdW$,

where W is a Wiener process.

181

Example 12.6

Assume that the cash position of a company, measured in thousands of dollars, follows a generalized Wiener process with a drift of 20 per year and a variance rate of 900 per year. Initially, the cash position is 50. Find the probabilities of negative cash positions after 1 year and after 3 months.

182

Definition 12.7

An Itô process is a generalized Wiener process in which the parameters a and b are functions of X and t.

183

Remark 12.8

Now we discuss the stochastic process usually assumed for the price S of a non-dividend-paying stock. A generalized Wiener process $dS=adt+bdW$ is not appropriate as

- expected percentage change of S should remain constant, not its expected absolute change,
- uncertainty as to the size of future stock price movements should be proportional to the level of the stock price.

184

Remark 12.8 (continued)

So we are using the Itô process

$$dS=\mu Sdt+\sigma SdW,$$

where μ is the expected return and σ is the volatility of the stock price. This model can be regarded as the limiting case of the random walk represented by binomial trees as the time step becomes smaller. The model is also known as geometric Brownian motion.

185

Example 12.9

Consider a non-dividend-paying stock with volatility 30% (pa) providing expected return of 15%. Suppose the stock price is initially 100. Assuming the stock price follows GBM, what is the probability that the stock price after one week is more than 100?

186

Example 12.10

Here we discuss Monte-Carlo simulation. Let $\mu=0.14$, $\sigma=0.2$, $\Delta t=0.01$.

t	S(t)	ε	ΔS(t)
0	20.000	0.52	
0.01		1.44	
0.02		-0.86	
0.03		1.46	
0.04		-0.69	
0.05		-0.74	
0.06		0.21	
0.07		-1.1	
0.08		0.73	
0.09		1.16	
0.1		2.56	

187

Theorem 12.11

(Itô's lemma)

If dW is a WP,

$dX=a(X,t)dt+b(X,t)dW$,

and $G=G(x,t)$, then

$dG=(G_x a+G_t+G_{xx}b^2/2)dt+G_x bdW$.

Corollary 12.12

If $dS=\mu Sdt+\sigma SdW$ and $G=G(s,t)$, then

$dG=(G_s\mu S+G_t+G_{ss}\sigma^2S^2/2)dt$
$+G_s\sigma SdW$.

Corollary 12.13

If $dS=\mu Sdt+\sigma SdW$ and F is the forward price of a forward contract on the non-dividend-paying stock, then

$dF=(\mu-r)Fdt+\sigma FdW$.

Remark 12.14

Note that F in Corollary 12.13 follows again geometric Brownian motion with the same variance rate as the stock price and a growth rate equal to the excess return of the stock price over the risk-free rate.

Corollary 12.15

If $dS=\mu Sdt+\sigma SdW$ and $G=\ln(S)$, then

$dG=(\mu-\sigma^2/2)dt+\sigma dW$.