Chapter 13

The Black-Scholes-Merton Model

Definition 13.1

A nonnegative random variable X is said to have a lognormal distribution with parameters μ and σ if

$$\ln(X) \sim N(\mu, \sigma).$$

Example 13.2

- If S is a stock price following geometric Brownian motion, then S_T has a lognormal distribution, namely

$$\ln(S_T) \sim N(\ln(S_0) + (\mu - \sigma^2/2)T, \sigma T^{1/2}).$$

Example 13.2 (continued)

- Consider a stock with an initial price of \$40, an expected return of 16% (pa), and a volatility of 20% (pa). Find a 95%-confidence interval for $S_{1/2}$.

Theorem 13.3

If X has a lognormal distribution with parameters μ and σ, then

$$E(X) = e^{\mu + \sigma^2/2}$$

and

$$Var(X) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1).$$

Example 13.4

- If S is a stock price following geometric Brownian motion, then

$$E(S_T) = S_0 e^{\mu T}$$

and

$$Var(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1).$$

Example 13.4 (continued)

- Consider a stock with current price of $20, an expected return of 20% (pa), and a volatility of 40% (pa). Find the expected value and the variance of the stock price in one year.

199

Remark 13.5

Here we discuss how to estimate volatility.

200

Theorem 13.6

(Black-Scholes-Merton PDE)

If $dS=\mu S dt+\sigma S dW$ and f is the price of a call, then

$$f_t+rSf_s+\sigma^2S^2f_{ss}/2=rf.$$

201

Remark 13.7

- (BSM) has many solutions, but we are looking for a solution that satisfies the boundary condition

$$f=(S-K)^+ \text{ when } t=T$$

for a European call, or

$$f=(K-S)^+ \text{ when } t=T$$

for a European put.

202

Remark 13.7 (continued)

- The portfolio from the proof of Theorem 13.6 is not permanently riskless, only during Δt. To keep the portfolio riskless, frequent adjustments are to be made.

203

Remark 13.7 (continued)

- Any f that satisfies (BSM) is called a price of a tradeable derivative.

204

Example 13.8

- $f(S,t)=S-Ke^{-r(T-t)}$ is a price of a tradeable derivative.
- $f(S,t)=e^S$ is not a price of a tradeable derivative.
- $f(S,t)=e^{(\sigma^2-2r)(T-t)}/S$ is a price of a tradeable derivative.

Theorem 13.9

(Black-Scholes pricing formulas)

The prices at time 0 of a European call and put on a non-dividend-paying stock are

$$c=S_0N(d_1)-Ke^{-rT}N(d_2)$$

and

$$p=Ke^{-rT}N(-d_2)-S_0N(-d_1).$$

Theorem 13.9 (continued)

Here, N is the cdf of the standard normal distribution and

$$d_1=(\ln(S_0/K)+(r+\sigma^2/2)T)/(\sigma T^{1/2})$$

and

$$d_2=(\ln(S_0/K)+(r-\sigma^2/2)T)/(\sigma T^{1/2}).$$

Remark 13.10

- Let $S_0 \to \infty$
- Let $\sigma \to 0$
- For N we can use the polynomial approximation providing 6-decimal-place accuracy $N(x)=1-N'(x)(a_1k+a_2k^2+a_3k^3+a_4k^4+a_5k^5)$ if $x\geq 0$ and $N(x)=1-N(-x)$ if $x<0$. Here $k=1/(1+x\eta)$, $\eta=0.2316419$, $a_1=0.319381530$, $a_2=-0.356563782$, $a_3=1.781477937$, $a_4=-1.821255978$, $a_5=1.330274429$.

Example 13.11

$T=1/2$, $S_0=42$, $K=40$, $r=0.1$, $\sigma=0.2$. Find the prices of a European call and a European put.

Remark 13.12

We can also use risk-neutral valuation to prove BSPF:

- Assume that the expected return from the stock price is the risk-free rate.
- Calculate the expected payoff from the option.
- Discount at the risk-free rate.

Theorem 13.13

Let X be lognormally distributed with $\ln X \sim N(m,w)$. Then for $K>0$,

$$E((X-K)^+)=E(X)N(d_1)-KN(d_2),$$

where

$d_1=(\ln(E(X)/K)+w^2/2)/w$,
$d_2=(\ln(E(X)/K)-w^2/2)/w$.