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Chapter 4

Interest Rates

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Remark 4.1

Types of rates are:

- Treasury rates (government, virtually risk free)
- LIBOR rates (1/3/6/12-month in all major currencies, not totally risk free)
- Repo-rates (very little credit risk)

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Definition 4.2

Let $V(t)$ be the wealth at time t (years).
 We talk about **discrete** or **periodic compounding** with **frequency** m times a year and interest rate r per annum provided

$V(t) = V(0)(1+r/m)^{mt}$ for all $t \geq 0$.
 $(1+r/m)^{mt}$ is called the **growth factor**,
 $(1+r/m)^{-mt}$ is called the **discount factor**.

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Example 4.3

- Let $r=0.1$. Find the value of \$100 after 1 year with periodic compounding and $m=1, 2, 4, 12, 52, 365$.
- How long does it take to double a capital attracting interest at 6% daily?
- What is r if a deposit subject to annual compounding is doubled after 10 years?

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Definition 4.4

An **annuity** is a sequence of finitely many payments of a fixed amount due at equal time intervals.

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Proposition 4.5

For discrete annual compounding with rate r and payments of C every year, the present value of an annuity for n years is

$C(1-(1+r)^{-n})/r$.

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Example 4.6

Consider a loan of \$1000 to be paid back in 5 equal installments due at yearly intervals. The installments include both the interest payable each year calculated at 15% of the current outstanding loan and the repayment of a fraction of the loan (**amortized loan**).

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Example 4.7

Suppose that you took a mortgage of \$100,000 on a house to be paid back in 10 equal annual payments ($r=6\%$). If you decided to clear the mortgage after 8 years, how much would you need to pay on top of the 8th installment?

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Definition 4.8

A **perpetuity** is an infinite sequence of equal payments due at equal time intervals.

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Proposition 4.9

For discrete annual compounding with rate r and payments of C every year, the present value of a perpetuity is

C/r .

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Definition 4.10

We talk about **continuous compounding** at rate r provided

$V(t)=V(0)e^{rt}$ for all $t \geq 0$.

e^{rt} is called the **growth factor**,

e^{-rt} is called the **discount factor**.

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Remark 4.11

Under continuous compounding, the rate of growth of the wealth is proportional to the wealth:

$V'(t)=rV(t)$.

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Example 4.12

How long will it take to earn \$1 if $r=0.1$ (c.c.) and $V(0)=\$1$ million?

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Definition 4.13

- Two compounding methods are called **equivalent** if the corresponding growth factors over a period of one year are the same.
- If one of the growth factors is bigger, then that method is called **preferable**.

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Example 4.14

- What is the equivalent continuous rate for 10% semiannual compounding?
- What is the equivalent quarterly rate for 8% continuous compounding?

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Definition 4.15

For a given compounding method, the **effective rate** r_e is the rate for annual compounding equivalent to that method.

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Example 4.16

What is the effective rate for semiannual compounding with $r=10\%$?

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Definition 4.17

A **zero-coupon bond** involves a single payment, and the issuing institution promises to exchange the bond for its **face value** (**principal value**) at a given **maturity date**.

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Example 4.18

- Suppose a bond has face value $F=100$ and matures in 1 year. If $r=12\%$ (a.c.), find the present value of the bond.
- Find the interest rates for annual, semiannual, and continuous compounding implied by a **unit bond** with maturity 1 and value 0.9455 after half a year.

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Definition 4.19

A **coupon bond** promises a sequence of payments, consisting of the face value paid at maturity and coupons paid regularly, the last coupon being due at maturity.

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Example 4.20

Consider a bond with $F=100$, $T=5$, $C=10$ paid annually, $r=0.12$ continuously compounded. Find the value of this bond at times 0, 1, and 4.

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Proposition 4.21

For coupons paid annually and continuous compounding with constant rate r , the price of a bond with coupon value C , face value F , and maturity T years is

$$C(1-e^{-rT})/(e^r-1)+Fe^{-rT}.$$

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Definition 4.22

- Assuming that coupons are paid annually, $i=C/F$ is called the **coupon rate**.
- If the price of a bond is equal to its face value, we say the bond sells **at par**.
- The coupon rate that causes the bond to sell at par is called the **par yield**.

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Proposition 4.23

Assume that coupons are paid annually and interest rates are constant. Then the par yield is equal to the effective rate.

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Definition 4.24

The **bond yield** is the discount rate that, when applied to all cash flows, gives a bond price equal to its market price.

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Example 4.25

Suppose a 2-year Treasury bond with $F=100$ provides coupons at rate of 6% p.a. semiannually.

Maturity (years)	Treasury zero rate (%) c.c.
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

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Example 4.26

In this example we discuss the most popular approach to calculate Treasury zero rates from the prices of Treasury bonds, the **bootstrap method**.

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Example 4.26 (continued)

Bond principal (\$)	Time to maturity (years)	Annual coupon (\$)	Bond price (\$)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

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Definition 4.27

The **forward rate** is the future zero rate implied by today's term structure of zero interest rates.

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Example 4.28

Find the forward rates for the n th year (% p.a.).

Year (n)	Zero rate for an n-year investment (% p.a.)
1	3.0
2	4.0
3	4.6
4	5.0
5	5.3

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Proposition 4.29

Assume R_1 and R_2 are the zero rates for maturities T_1 and T_2 . Then the forward rate R_F between T_1 and T_2 is given by

$$R_F = (R_2 T_2 - R_1 T_1) / (T_2 - T_1).$$

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Definition 4.30

A **forward rate agreement (FRA)** is an agreement that a certain rate will apply to a certain principal during a certain future period.

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Example 4.31

Suppose company X enters into an FRA with Y that specifies that it will receive a fixed rate of $R_K=4\%$ on a principal of $L=1$ million for a 3-month period starting in 3 years. The actual 3-month LIBOR proves to be $R_M=4.5\%$. Find the cash flow to Y.

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Example 4.32

Suppose rates are as Example 4.28. Consider an FRA where we will receive $R_K=6\%$ (annual compounding) on $L=1$ million between times 1 and 2. Note $R_F=5\%$ is the forward rate calculated today. Find the present value of the FRA.

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Definition 4.33

The **duration** of a bond with price B and yield y that provides cash flow c_i at time t_i , $1 \leq i \leq n$, is defined by

$$D = \sum_{i=1}^n t_i c_i e^{-y t_i} / B$$

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Remark 4.34

- A zero-coupon bond has duration $t_1=T$
- Duration is a measure of how long on average the holder has to wait before receiving cash payments
- D is a convex combination of payment times
- Express ΔB in terms of D

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Example 4.35

Consider a 3-year 10% coupon bond (paid semiannually) with $F=100$, $y=0.12$ cc. Find the new bond price if the yield increases by ten **basis points**.

