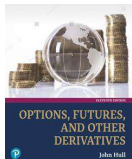
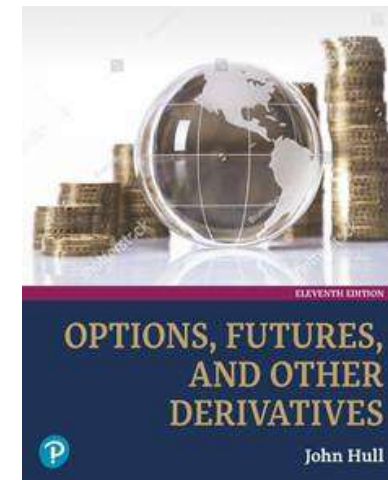


Math 5737/Econ 5337 Financial Mathematics

Martin Bohner
Fall Semester 2022

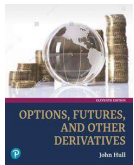


Math 5737/Econ 5337, Fall 2022



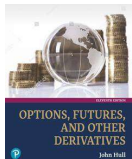
Literature

- Options, Futures, and other Derivatives, John C. Hull, Prentice Hall, 2022 (11th edition)
- Mathematics of Finance, Marek Capiński and Tomasz Zastawniak, Springer, 2011 (2nd edition)
- Risk-Neutral Valuation, Nick Bingham and Rüdiger Kiesel, Springer, 2004 (2nd edition)



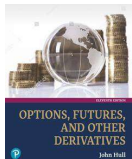
Chapter 1

Introduction



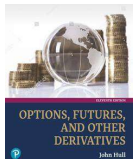
Definition 1.1

A derivative (derivative security, contingent claim) is a financial instrument whose value depends on (derives from, is contingent on) the values of other, more basic, underlying variables (the underlying).



Example 1.2

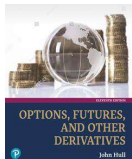
- Forwards
- Futures
- Swaps
- Options



Remark 1.3

Market forms for derivatives are

- Exchange-traded markets, e.g.,
NYSE 1792, CBOT 1848, CME 1919,
NASDAQ 1971, CBOE 1973
 - open outcry system
 - electronic
- Over-the-counter markets



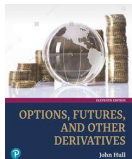
Definition 1.4

There are three types of traders:

- **Hedgers** attempt to reduce exposure to risk a company already faces
- **Speculators** invest available funds opportunistically in the hope of making a profit
- **Arbitrageurs** try to lock in riskless profit

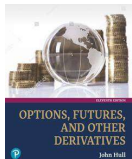
arbitrage: risk-free profit with no initial investment

no-arbitrage principle: arbitrage opportunities are absent



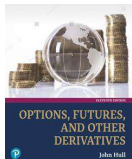
Definition 1.5

- A **forward contract** is an agreement to sell or buy an asset at a fixed date in the future (**delivery time**) for a price specified in advance (**forward price, delivery price**).
- The party selling the asset assumes what is termed a **short (forward) position**, while the party buying the asset enters into a **long (forward) position**.



Definition 1.5 (continued)

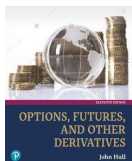
- It costs nothing to enter a forward contract. A forward contract can be contrasted to a **spot contract**, which is an agreement to buy or sell an asset today.



Example 1.6 (a)

Suppose that

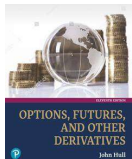
- The spot price of $\frac{1}{4}$ oz gold is \$300
- The 1-year forward price of $\frac{1}{4}$ oz gold is \$340
- The 1-year interest rate is 5% p.a.



Example 1.6 (b)

Suppose that

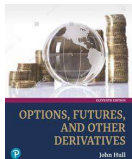
- The spot price of $\frac{1}{4}$ oz gold is \$300
- The 1-year forward price of $\frac{1}{4}$ oz gold is \$310
- The 1-year interest rate is 5% p.a.



Example 1.7

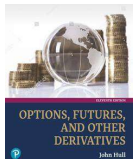
Foreign exchange quote for GBP in USD
on August 14, 2022

	buy	sell
spot	1.21152	1.21172
1-month forward	1.21219	1.21245
3-month forward	1.21393	1.21432
6-month forward	1.21719	1.21746



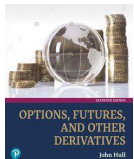
Example 1.7 (continued)

- (a) Say it is August 14, 2022, and Import Co, a company based in the US, knows that it will have to pay GBP 10 million on November 14, 2022, for goods purchased from a British supplier.
- (b) Export Co exports goods to UK and knows it will receive GBP 10 million in three months.



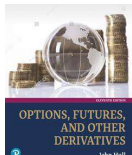
Definition 1.8

- A **European call option** is a contract giving the holder the right (but no obligation) to buy an asset (**the underlying**) for a price fixed in advance (**exercise price, strike price**) at a specified future time (**exercise time, expiry time, maturity**).
- A **European put option** is a contract giving the right to sell an asset for a certain strike price at a certain exercise time.



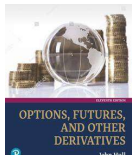
Definition 1.8 (continued)

- An **American put** or **call option** can be exercised any time up to and including the **expiry time**.
- Since payoffs are nonnegative, a premium (the market price of the option) must be paid when acquiring an option.



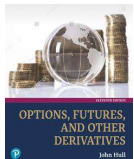
Example 1.9

Suppose an investor owns 1000 Microsoft shares on August 12, 2022, \$292 per share is the current price. Suppose the investor is concerned about a possible share price decline in the next 2 months and wants protection. The investor could buy 10 put option contracts with strike price \$290 and expiration time October 21, 2022. Assume each option price is \$14. Ignore the time value of money.



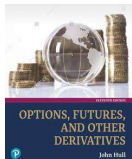
Example 1.10

On August 21, 2022, European calls on Twitter stock with $K=40$ USD and $T=$ October 21, 2022 were traded at 6.50 USD at NASDAQ. Ignoring the time value of money, when will the investment bring profit?



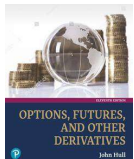
Chapter 2

Mechanics of Futures Markets



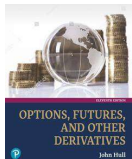
Definition 2.1

A **futures contract** involves an underlying and a specified delivery time. In addition to the stock prices S_i , the market dictates **futures prices** F_i , which are random variables. While a long forward contract involves just a single cash flow $S_T - K$ at delivery time T , a futures contract involves a random cash flow $F_i - F_{i-1}$, known as **marking to market**:



Definition 2.1 (continued)

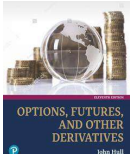
Each investor entering a futures contract has to pay a deposit, the **initial margin**. In a long futures position, $F_i - F_{i-1}$ is added to the deposit at time i . Any excess above the initial margin may be withdrawn by the investor. However, if deposit drops below the **maintenance margin**, the clearing house will issue a **margin call**, requesting the investor to restore the deposit to the level of the initial margin. If the investor fails to respond to a margin call, the clearing house will close the futures position. It costs nothing to enter, close, or alter a futures contract.



Example 2.2

Assume initial margin and maintenance margin are 10% and 5%, respectively, of the futures price, in a long futures position.

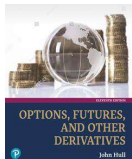
i	F_i	cash flow	Deposit at beginning of Day i	Payment into account	Deposit at end of Day i
0	140				
1	138				
2	130				
3	140				
4	150				



Remark 2.3

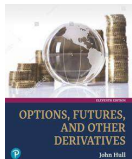
Comparison of forward and futures contracts

Forwards	Futures
Over-the-counter market	Exchange-traded market
Not standardized	Standardized contract
Settled at end of contract	Settled daily
Delivery takes always place	Contract is usually closed out prior to maturity
Some credit risk	Virtually no credit risk



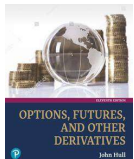
Chapter 3

Hedging Strategies using Futures



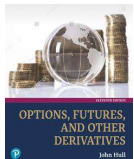
Definition 3.1

- A **short hedge** is a hedge involving a short position in a futures contract
- A **long hedge** is a hedge involving a long position in a futures contract



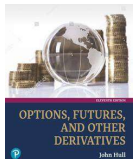
Remark 3.2

- Short hedges are appropriate when you know you will sell the asset in the future
- Long hedges are appropriate when you know you will buy the asset in the future



Remark 3.2 (continued)

- Arguments for hedging: Minimizing risks arising from interest rates, exchange rates, and other market variables
- Arguments against hedging: Shareholders can do it themselves, risk may increase if competitors do not hedge, explaining a situation where there is a loss on the hedge and a gain on the underlying can be difficult

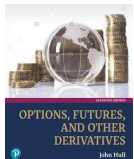


Definition 3.3

- The **basis** in a hedging situation is defined by

$$b_i = S_i - F_i$$

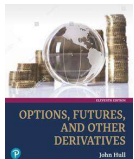
- If b_i is increasing, we say that this is **strengthening** of the basis
- If b_i is decreasing, this is called **weakening** of the basis



Example 3.4

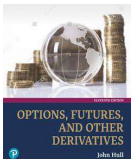
Assume a hedge is put in place at time 1 and closed out at time 2, $S_1=2.50$, $F_1=2.20$, $S_2=2.00$, $F_2=1.90$

- If hedger knows that asset will be sold at time 2, what is the effective price that is obtained for the asset with hedging?
- If hedger knows that asset will be purchased at time 2, what is the effective price that is paid for the asset with hedging?



Definition 3.5

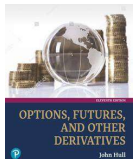
- **Cross hedging** occurs if the asset underlying the futures price is not the same as the asset whose price is being hedged.
- The **hedge ratio** is the ratio of the size of futures contracts to the size of shares of the underlying held.
- The **minimum variance hedge ratio** h^* is the hedge ratio that results in the risk (measured by the variance) being minimal.



Theorem 3.6

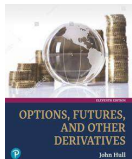
$$h^* = \rho_{SF} \sigma_S / \sigma_F$$

where σ_S and σ_F are the standard deviations of ΔS and ΔF , respectively, and ρ_{SF} is the correlation coefficient between ΔS and ΔF .



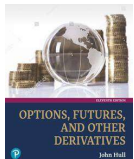
Example 3.7

An airline expects to purchase 2 million gallons of jet fuel in one month and decides to use heating oil futures for hedging. One futures contract is on 42,000 gallons. It is given that the standard deviations of the changes in fuel price and in the futures price (per gallon) are 0.03 and 0.04, respectively, and the correlation coefficient between these two changes is 0.9.



Example 3.8

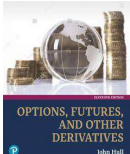
An airline expects to purchase 2 million gallons of jet fuel in one month and decides to use heating oil futures for hedging. One futures contract is on 42,000 gallons.



Example 3.8 (continued)

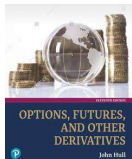
Month i	Change in fuel price per gallon x_i	Change in futures price per gallon y_i
1	0.029	0.021
2	0.020	0.035
3	-0.044	-0.046
4	0.008	0.001
5	0.026	0.044
6	-0.019	-0.029
7	-0.010	-0.026

8	-0.007	-0.029
9	0.043	0.048
10	0.011	-0.006
11	-0.036	-0.036
12	-0.018	-0.011
13	0.009	0.019
14	-0.032	-0.027
15	0.023	0.029



Example 3.9

A **stock index** tracks changes in the value of a hypothetical portfolio of stocks. Futures contracts on stock indices are settled in cash.



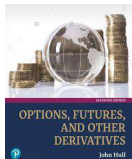
Example 3.9 (continued)

The **beta** β of a portfolio is defined by

$$\beta_V = \rho_{VM} \sigma_V / \sigma_M$$

and it satisfies (Capital Asset Pricing Model)

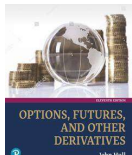
$$\beta = (\mu_V - r_F) / (\mu_M - r_F).$$



Example 3.9 (continued)

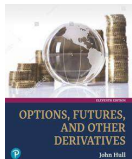
We assume now:

- Value of S&P500 index: 1000
- Value of portfolio: 5,000,000
- Risk-free interest rate: $r_f=4\%$ p.a.
- Dividend yield on index: 1% p.a.
- Beta of portfolio: $\beta=1.5$
- One futures contract is \$1010 and for delivery of 250 times the index



Chapter 4

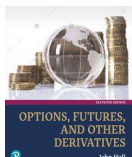
Interest Rates



Remark 4.1

Types of rates are:

- Treasury rates (government, virtually risk free)
- LIBOR rates (1/3/6/12-month in all major currencies, not totally risk free)
- Repo-rates (very little credit risk)



Definition 4.2

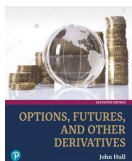
Let $V(t)$ be the wealth at time t (years).

We talk about **discrete** or **periodic compounding** with **frequency** m times a year and interest rate r per annum provided

$$V(t) = V(0)(1+r/m)^{mt} \text{ for all } t \geq 0.$$

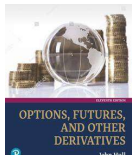
$(1+r/m)^{mt}$ is called the **growth factor**,

$(1+r/m)^{-mt}$ is called the **discount factor**.



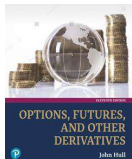
Example 4.3

- Let $r=0.1$. Find the value of \$100 after 1 year with periodic compounding and $m=1, 2, 4, 12, 52, 365$.
- How long does it take to double a capital attracting interest at 6% daily?
- What is r if a deposit subject to annual compounding is doubled after 10 years?



Definition 4.4

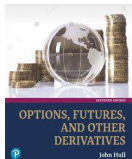
An **annuity** is a sequence of finitely many payments of a fixed amount due at equal time intervals.



Proposition 4.5

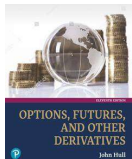
For discrete annual compounding with rate r and payments of C every year, the present value of an annuity for n years is

$$C(1-(1+r)^{-n})/r.$$



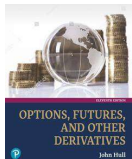
Example 4.6

Consider a loan of \$1000 to be paid back in 5 equal installments due at yearly intervals. The installments include both the interest payable each year calculated at 15% of the current outstanding loan and the repayment of a fraction of the loan (**amortized loan**).



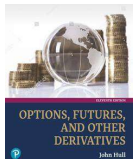
Example 4.7

Suppose that you took a mortgage of \$100,000 on a house to be paid back in 10 equal annual payments ($r=6\%$). If you decided to clear the mortgage after 8 years, how much would you need to pay on top of the 8th installment?



Definition 4.8

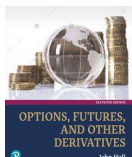
A **perpetuity** is an infinite sequence of equal payments due at equal time intervals.



Proposition 4.9

For discrete annual compounding with rate r and payments of C every year, the present value of a perpetuity is

$$C/r.$$



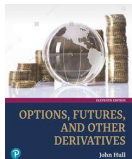
Definition 4.10

We talk about **continuous compounding** at rate r provided

$$V(t) = V(0)e^{rt} \text{ for all } t \geq 0.$$

e^{rt} is called the **growth factor**,

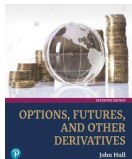
e^{-rt} is called the **discount factor**.



Remark 4.11

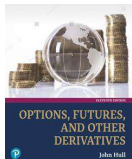
Under continuous compounding, the rate of growth of the wealth is proportional to the wealth:

$$V'(t) = rV(t).$$



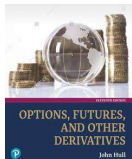
Example 4.12

How long will it take to earn \$1 if $r=0.1$ (c.c.) and $V(0)=\$1$ million?



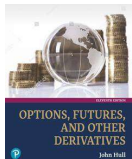
Definition 4.13

- Two compounding methods are called **equivalent** if the corresponding growth factors over a period of one year are the same.
- If one of the growth factors is bigger, then that method is called **preferable**.



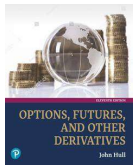
Example 4.14

- What is the equivalent continuous rate for 10% semiannual compounding?
- What is the equivalent quarterly rate for 8% continuous compounding?



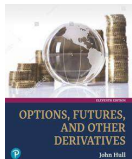
Definition 4.15

For a given compounding method, the **effective rate** r_e is the rate for annual compounding equivalent to that method.



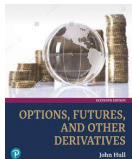
Example 4.16

What is the effective rate for semiannual compounding with $r=10\%$?



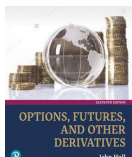
Definition 4.17

A **zero-coupon bond** involves a single payment, and the issuing institution promises to exchange the bond for its **face value** (**principal value**) at a given **maturity date**.



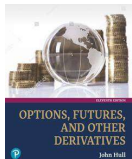
Example 4.18

- Suppose a bond has face value $F=100$ and matures in 1 year. If $r=12\%$ (a.c.), find the present value of the bond.
- Find the interest rates for annual, semiannual, and continuous compounding implied by a **unit bond** with maturity 1 and value 0.9455 after half a year.



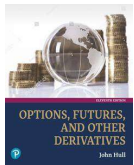
Definition 4.19

A **coupon bond** promises a sequence of payments, consisting of the face value paid at maturity and coupons paid regularly, the last coupon being due at maturity.



Example 4.20

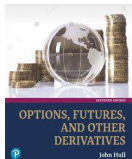
Consider a bond with $F=100$, $T=5$, $C=10$ paid annually, $r=0.12$ continuously compounded. Find the value of this bond at times 0, 1, and 4.



Proposition 4.21

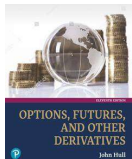
For coupons paid annually and continuous compounding with constant rate r , the price of a bond with coupon value C , face value F , and maturity T years is

$$C(1-e^{-rT})/(e^r-1)+Fe^{-rT}.$$



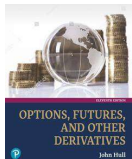
Definition 4.22

- Assuming that coupons are paid annually, $i=C/F$ is called the **coupon rate**.
- If the price of a bond is equal to its face value, we say the bond sells **at par**.
- The coupon rate that causes the bond to sell at par is called the **par yield**.



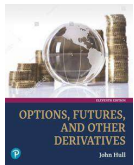
Proposition 4.23

Assume that coupons are paid annually and interest rates are constant. Then the par yield is equal to the effective rate.



Definition 4.24

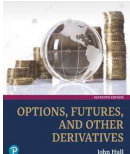
The **bond yield** is the discount rate that, when applied to all cash flows, gives a bond price equal to its market price.



Example 4.25

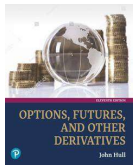
Suppose a 2-year Treasury bond with $F=100$ provides coupons at rate of 6% p.a. semiannually.

Maturity (years)	Treasury zero rate (%) c.c.
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8



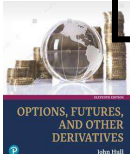
Example 4.26

In this example we discuss the most popular approach to calculate Treasury zero rates from the prices of Treasury bonds, the **bootstrap method**.



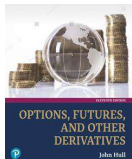
Example 4.26 (continued)

Bond principal (\$)	Time to maturity (years)	Annual coupon (\$)	Bond price (\$)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6



Definition 4.27

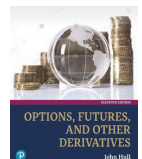
The **forward rate** is the future zero rate implied by today's term structure of zero interest rates.



Example 4.28

Find the forward rates for the n th year (% p.a.).

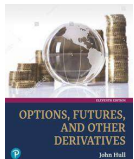
Year (n)	Zero rate for an n-year investment (% p.a.)
1	3.0
2	4.0
3	4.6
4	5.0
5	5.3



Proposition 4.29

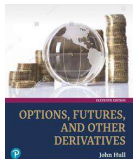
Assume R_1 and R_2 are the zero rates for maturities T_1 and T_2 . Then the forward rate R_F between T_1 and T_2 is given by

$$R_F = (R_2 T_2 - R_1 T_1) / (T_2 - T_1).$$



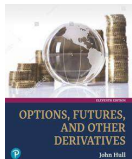
Definition 4.30

A **forward rate agreement (FRA)** is an agreement that a certain rate will apply to a certain principal during a certain future period.



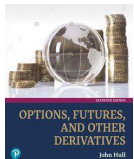
Example 4.31

Suppose company X enters into an FRA with Y that specifies that it will receive a fixed rate of $R_K=4\%$ on a principal of $L=1$ million for a 3-month period starting in 3 years. The actual 3-month LIBOR proves to be $R_M=4.5\%$. Find the cash flow to Y.



Example 4.32

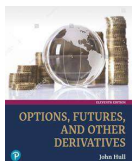
Suppose rates are as Example 4.28. Consider an FRA where we will receive $R_K=6\%$ (annual compounding) on $L=1$ million between times 1 and 2. Note $R_F=5\%$ is the forward rate calculated today. Find the present value of the FRA.



Definition 4.33

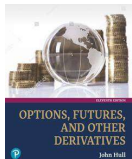
The **duration** of a bond with price B and yield y that provides cash flow c_i at time t_i , $1 \leq i \leq n$, is defined by

$$D = \sum_{i=1}^n t_i c_i e^{-y t_i} / B$$



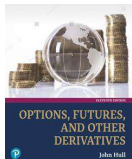
Remark 4.34

- A zero-coupon bond has duration $t_1=T$
- Duration is a measure of how long on average the holder has to wait before receiving cash payments
- D is a convex combination of payment times
- Express ΔB in terms of D



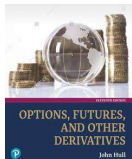
Example 4.35

Consider a 3-year 10% coupon bond (paid semiannually) with $F=100$, $y=0.12$ cc. Find the new bond price if the yield increases by ten **basis points**.



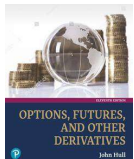
Chapter 5

Forward and Futures Prices



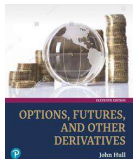
Definition 5.1

- An **investment asset** is an asset which is held by a significant number of people for investment purposes
- A **consumption asset** is held primarily for consumption purposes



Remark 5.2

Short selling involves selling an asset that is not owned. The broker borrows the asset from another client and sells it. At some stage the asset must be replaced. Until then, dividends or other benefits must be paid to the owner and a margin account must be maintained.



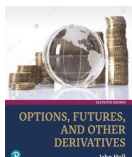
Theorem 5.3

For an investment asset providing no income, we have

$$F_0 = F(0, T) = S_0 e^{rT}$$

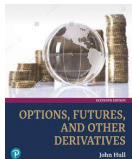
and

$$F(t, T) = S(t) e^{r(T-t)}.$$



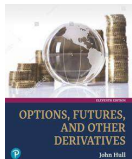
Example 5.4

Consider a long forward contract to purchase a non-dividend-paying stock in 3 months. Assume $S_0=40$ and $r=5\%$ cc. Find F_0 .



Example 5.5

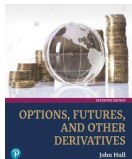
Suppose the stock price on 1 April 2022 is 10% lower than that on 1 January 2022 and $r=6\%$. What is the percentage drop of the forward price on 1 April 2022 compared to that on 1 January 2022 for a forward with delivery on 1 October 2022?



Theorem 5.6

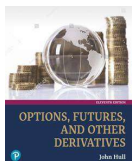
For an investment asset providing income with present value I , we have

$$F_0 = (S_0 - I)e^{rT}.$$



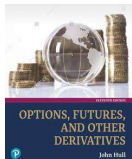
Example 5.7

Consider a 10-month forward on a stock with $S_0=50$, $r=8\%$, that pays dividends of \$0.75 after 3, 6, and 9 months. Find F_0 .



Example 5.8

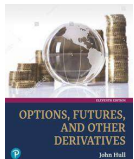
Consider a long 9-month forward to purchase a coupon-bearing bond with current price \$900. The coupon payment is \$40 after 4 months. Suppose the 4-month/9-month risk-free rates are 3%/4%. Find F_0 .



Theorem 5.9

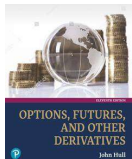
For an investment asset providing income continuously at a known yield q (cc), we have

$$F_0 = S_0 e^{(r-q)T}.$$



Example 5.10

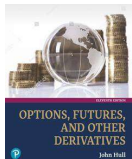
Consider a forward on a stock paying a continuous dividend at rate 3%, $T=1/2$ year, $S_0=98.34$, $r=10\%$. Find F_0 .



Theorem 5.11

The value of a long forward contract at time t is

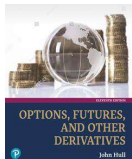
$$V(t) = [F(t, T) - F(0, T)]e^{-r(T-t)}.$$



Corollary 5.12

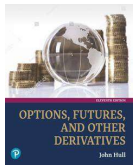
The value of a long forward on an investment asset providing

- no income is $S(t) - F_0 e^{-r(T-t)}$
- known income with present value I is $S(t) - I(t) - F_0 e^{-r(T-t)}$
- known yield at rate q is $S(t) e^{-q(T-t)} - F_0 e^{-r(T-t)}$.



Example 5.13

A long forward contract on a non-dividend-paying stock was entered into some time ago. It currently has 6 months to maturity. The stock price is \$25, the delivery price \$24, $r=0.1$. What is its value now?

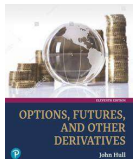


Remark 5.14

A **stock index** can be viewed as an investment asset paying a dividend yield. Thus, by Theorem 5.9, if q is the average dividend yield on the portfolio represented by the index during the life of a futures contract, then

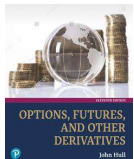
$$F_0 = S_0 e^{(r-q)T}$$

(otherwise, index arbitrage occurs).



Example 5.15

Consider a 3-month futures contract on the S&P 500. Suppose that the stocks underlying the index provide a dividend yield of 1% pa, the current value of the index is 800, and $r=6\%$ cc. Find F_0 .

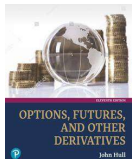


Remark 5.16

A **foreign currency** is analogous to a security providing a dividend yield, which is equal to the foreign interest rate r_f . Thus, if S_0 (in \$) is the spot exchange rate and r is the US interest rate, then the forward or futures price on a foreign currency is

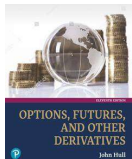
$$F_0 = S_0 e^{(r-r_f)T}$$

(this is also known as the **interest rate parity** relationship from International Finance).



Example 5.17

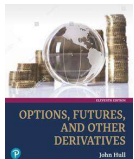
Suppose 2-year interest rates in Australia and the US are 5% and 7%, respectively, cc. Assume $S_0=0.62$ USD per AUD. Find F_0 .



Remark 5.18

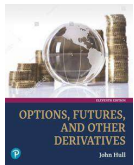
Storage costs can be treated as negative income with present value U or negative yield u . Hence, for commodities that are investment assets, we have

$$F_0 = (S_0 + U)e^{rT} \quad \text{or} \quad F_0 = S_0 e^{(r+u)T}.$$



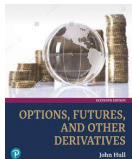
Example 5.19

Consider a 1-year futures on an investment asset providing no income, but it costs \$2 (at the end of the year) to store the asset. Assume $S_0=450$ and $r=7\%$ cc. Find F_0 .



Chapter 6

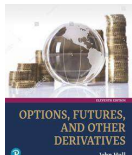
Interest Rate Futures



Definition 6.1

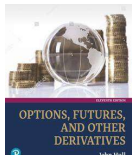
Daycount conventions in the US are given as X/Y , where X and Y define how to count the number of days between two dates and in the reference period, respectively, and are

- actual/actual (e.g., for US Treasury bonds)
- 30/360 (e.g., for US corporate and municipal bonds)
- actual/360 (e.g., for US money market instruments).



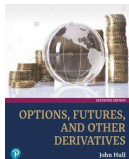
Example 6.2

(a) For a Treasury bond with $F=100$, coupon rate 8% with coupons paid on March 1 and September 1, 2022, find the interest earned between March 1 and July 3.



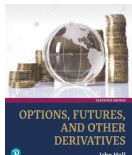
Example 6.2 (continued)

(b) For a corporate bond with the same data as in (a), answer the same question.



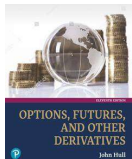
Definition 6.3

- Treasury bonds ($F=100$) and futures on them are **quoted** as $x-y$, which means that the **quoted price** equals $\$(x+y/32)$.
- The **cash price** (or **dirty price**) equals to the quoted price (or **clean price**) plus the accrued interest since the last coupon date.



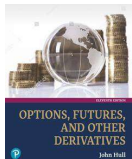
Example 6.4

Find the cash price of an 11% coupon Treasury bond with $F=100,000$ maturing on July 10, 2027 with a quote of 95-16 on March 5, 2022.



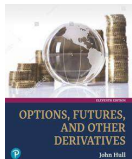
Remark 6.5

We discuss Treasury bond futures traded at CBOT. One contract involves delivery of \$100,000 face value of bonds. Any government bond with more than 15 years to maturity may be delivered by the party with the short futures position. Each bond has a **conversion factor** which is equal to the quoted price the bond would have per dollar of the principal on the first day of the delivery month assuming that interest for all maturities is 6% pa with semi-annual compounding.



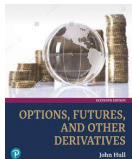
Remark 6.5 (continued)

All dates are rounded down to the nearest 3 months. If, after rounding, the bond lasts for an integer number of 6-month periods, the first coupon is assumed to be paid in 6 months. If not, the first coupon is assumed to be paid in 3 months and accrued interest is subtracted. Using the CF, cash received for each \$100 face value of bond delivered is $SP \times CF + AI$.



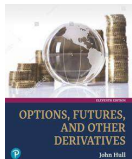
Example 6.6

If SP is 90-00, CF of the bond delivered is 1.3800, and AI on this bond at delivery is \$3 per \$100, then the party who is with short position in one futures contract would deliver bonds with face value \$100,000 and receives how much?



Example 6.7

- (a) Find the CF of a 10% coupon bond with 20 years and 2 months to maturity.
- (b) Find the CF of an 8% coupon bond with 18 years and 4 months to maturity.

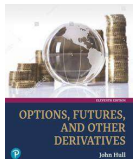


Definition 6.8

The **cheapest-to-deliver** bond is the one such that

$$(QP+AI)-(SP\times CF+AI)$$

is least.



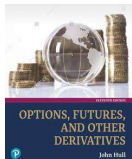
Example 6.9

Assume the most recent SP is 93-08 and the party with the short position has decided to deliver. Find the CTDB.

Bond	QP (\$)	CF
1	99.50	1.0382
2	143.50	1.5188
3	119.75	1.2615

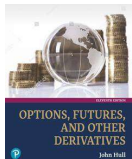
Example 6.10

Suppose that, in a Treasury bond futures contract, it is known that the CTDB will be a 12% coupon bond with $CF=1.4000$ and delivery in 270 days, $r=10\%$ pa cc, $QP=120$. The last coupon date was 60 days ago, the next one is in 122 days. Find the quoted price on the futures.



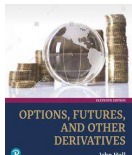
Chapter 7

Swaps



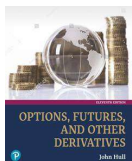
Definition 7.1

A **swap** is an agreement to exchange cash flows at specified future times according to certain specified rules.



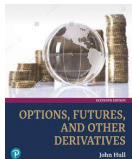
Definition 7.2

In a “plain vanilla” interest rate swap, a company agrees to pay cash flows equal to interest at a predetermined **fixed rate** on a **notional principal** for a number of years, while it receives interest at a **floating rate** on the same notional principal for the same period of time.



Example 7.3

Consider a 3-year swap between Microsoft and Intel initiated on Mar 5, 2022. MS agrees to pay INT 5% (sa) on \$100 million, while INT agrees to pay MS the 6-month LIBOR on the same principal (payments every 6 months). MS is the **fixed-rate payer**, while INT is the **floating-rate payer**.



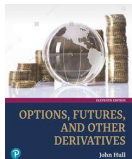
Example 7.4

(a) Interest rate swaps can be used to transform a liability

- from fixed rate to floating rate

- from floating rate to fixed rate.

Suppose MS has arranged to borrow \$100 million at LIBOR+10 basis points, INT at 5.2%.

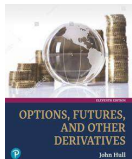


Example 7.4 (continued)

(b) Interest rate swaps can be used to convert an investment

- from fixed rate to floating rate
- from floating rate to fixed rate.

Suppose MS owns \$100 million in bonds providing 4.7% interest, INT has an investment of \$100 million yielding LIBOR minus 20 basis points.

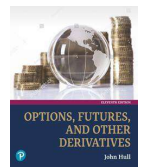


Example 7.5

- Here we discuss **comparative advantage**.

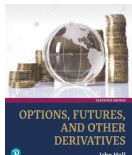
	Fixed	Floating
AAA Corp	4.0%	LIBOR +0.3%
BBB Corp	5.2%	LIBOR +1.0%

- Credit ratings
AAA, AA, A, BBB, BB, B, CCC



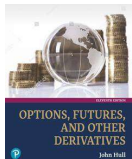
Remark 7.6

Often swaps have to be administered by a **financial intermediary** that will keep about 3-4 basis points.



Remark 7.7

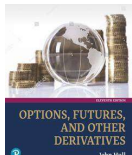
Interest rate swaps can be valued by regarding them as a difference of two bonds or as a portfolio of FRAs.



Example 7.8

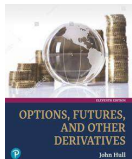
X pays 6-month LIBOR and receives 8% (sa) on \$100 million. The remaining life of the swap is 1.25 years, 3/9/15-month LIBOR are 10%, 10.5%, 11% (cc), and 6-month LIBOR at last payment date was 10.2% (sa).

- (a) Value the swap as a difference of bonds.
- (b) Value the swap as a portfolio of FRAs.



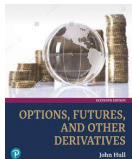
Definition 7.9

A **currency swap** (in its simplest form) involves exchanging principal and interest payments in one currency for principal and interest payments in another.



Example 7.10

Consider a 5-year currency swap between IBM and BP entered into on Feb 1, 2022. IBM pays a fixed rate of 7% in GBP and receives a fixed rate of 4% in USD from BP. Interest rate payments are made once a year and the principal amounts are \$15 and £10 million. This is a **fixed-for-fixed** currency swap.

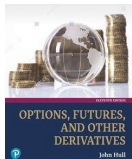


Example 7.11

- Here we discuss comparative advantage.

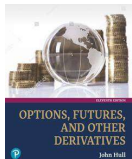
	USD	AUD
GM	5.0%	12.6%
Qantas	7.0%	13.0%

- Suppose GM wants to borrow 20 million AUD and Qantas wants to borrow 12 million USD, and the current rate is 0.6 USD per AUD.



Remark 7.12

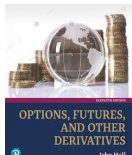
Currency swaps can be valued by regarding them as a difference of two bonds or as a portfolio of forward contracts.



Example 7.13

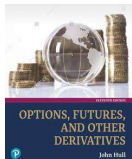
Suppose that LIBOR in Japan and US (both flat) are 4% and 9%, respectively (cc). An FI has entered into a currency swap in which it receives 5% in yen and pays 8% in USD once a year. The principals are \$10 million and 1200 million yen, and the current exchange rate is 110 yen for \$1. The swap will last for another 3 years.

- (a) Value the swap as a difference of bonds.
- (b) Value the swap as a portfolio of forward contracts.



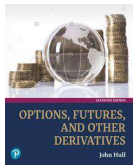
Chapter 8

Mechanics of Options Markets



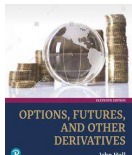
Definition 8.1

- A **call** is an option to buy
- A **put** is an option to sell
- To **buy** an option means to be in a long position
- To **write** (or **sell**) an option means to be in a short position



Example 8.2

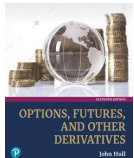
- Payoff from buying a European
 - Call
 - Put
- Payoff from writing a European
 - Call
 - Put
- Profit from positions in European options



Example 8.3

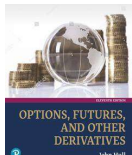
Suppose a stock is \$60, a European call with strike price \$70 costs \$1 and a European put with strike price \$50 costs \$3. Draw profit graphs for the following strategies (ignoring the time value of money) and determine for which stock prices the profit will be positive.

- (a) Long positions in a call and a put.
- (b) Long positions in three calls and two puts.
- (c) Long stock, long put, short call.
- (d) Long stock, two long puts, two short calls.



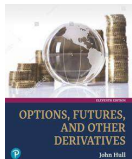
Remark 8.4

- Assets underlying exchange-traded options:
 - Stocks (~ 1000 different ones, usually 100 shares per contract)
 - Foreign currency (size depends on currency, e.g, £31,250; 6.25 million yen)
 - Stock indices (100×index, settlement in cash)
 - Futures (option matures just before delivery period for futures)



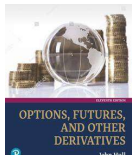
Remark 8.4 (continued)

- Stock options specifications:
 - Expiration date (Sat right after 3rd Fri of expiry month, 4:30 pm)
 - Strike price (spacing \$2.50/5/10)



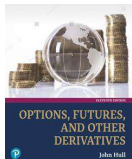
Definition 8.5

- We say that an option is **in the money** if
 - $K < S(t)$ for call
 - $K > S(t)$ for put
- and **out of the money** if
 - $K > S(t)$ for call
 - $K < S(t)$ for put
- and **at the money** if
 - $K = S(t)$.
- We may also refer to an option being **deep** in or out of the money.



Chapter 9

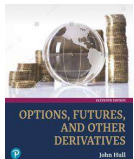
Properties of Stock Options



Proposition 9.1

If we let c , p , C , P be the prices of a European call, European put, American call, American put, respectively (all on the same stock, same K and same T), then

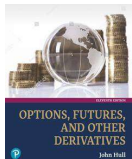
$$0 \leq c \leq C \text{ and } 0 \leq p \leq P.$$



Theorem 9.2

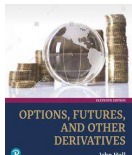
For a stock paying no dividends,

$$(S_0 - Ke^{-rT})^+ \leq c < S_0.$$



Example 9.3

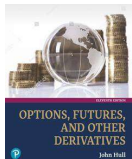
For a European call on a non-dividend paying stock with $S_0=51$, $K=50$, $T=1/2$, $r=0.12$, find the bounds on c .



Theorem 9.4

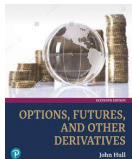
For a stock paying no dividends,

$$(Ke^{-rT} - S_0)^+ \leq p < Ke^{-rT}.$$



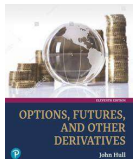
Example 9.5

For a European put on a non-dividend paying stock with $S_0=38$, $K=40$, $T=1/4$, $r=0.1$, find the bounds on p .



Remark 9.6

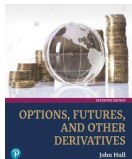
- Theorems 9.2 and 9.4 can be graphically summarized
- Note also that by Corollary 5.12 (first part), $S_0 - Ke^{-rT} = V(0)$ is the value of a long forward contract on the non-dividend paying stock



Theorem 9.7

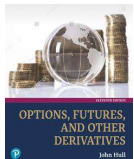
Put-call parity holds:

$$c - p = S_0 - Ke^{-rT}.$$



Example 9.8

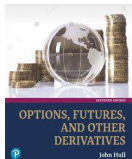
Suppose a stock paying no dividends trades at \$15.60 a share. European calls on the stock with strike price \$15 and exercise date in 3 months are trading at \$2.83, $r=6.72\%$ (cc). What is the price of a European put on the same stock with the same strike price and the same exercise date?



Theorem 9.9

For a stock paying no dividends,

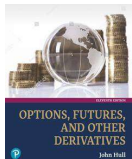
$$C=c.$$



Corollary 9.10

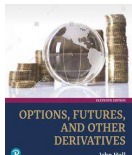
For a stock paying no dividends,

$$(S_0 - Ke^{-rT})^+ \leq C < S_0.$$



Example 9.11

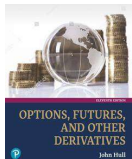
Suppose $S=10$, strike price of American put expiring in one year is \$80, and $r=0.16$. Here it is better to do early exercise.



Theorem 9.12

For a stock paying no dividends,

$$(K - S_0)^+ \leq P < K.$$

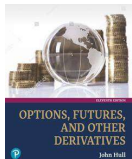


Theorem 9.13

Put-call parity estimates hold:

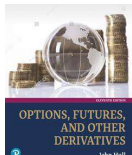
$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$$

(for a stock paying no dividends).



Example 9.14

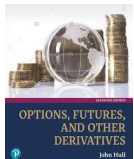
Suppose $S_0=19$, $K=20$, $T=5/12$,
 $r=0.1$, $C=1.50$. Find the bounds on
 P .



Theorem 9.15

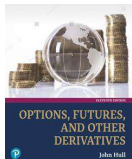
For a stock paying dividends with present value D ,

- $(S_0 - D - Ke^{-rT})^+ \leq c < S_0 - D$
- $(Ke^{-rT} + D - S_0)^+ \leq p < Ke^{-rT}$
- $c - p = S_0 - D - Ke^{-rT}$



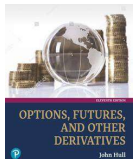
Theorem 9.15 (continued)

- $S_0 - D - K \leq C - P \leq S_0 - Ke^{-rT}$
- $\max\{0, S_0 - D - Ke^{-rT}, S_0 - K\} \leq C < S_0$
- $\max\{0, Ke^{-rT} + D - S_0, K - S_0\} \leq P < K$



Remark 9.16

In the rest of this chapter, we discuss the dependence of c , p , C , P on K , S , T , and r (varying only one variable while keeping the others constant; all on stocks paying no dividends).



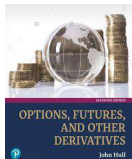
Proposition 9.17

c is nonincreasing in K while p is nondecreasing in K , i.e.,

$$c(K) \geq c(K') \text{ and } p(K) \leq p(K')$$

for

$$K \leq K'.$$



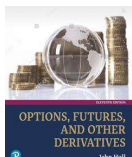
Theorem 9.18

c and p are Lipschitz continuous in K , namely,

$$|c(K) - c(K')| \leq e^{-rT} |K - K'|$$

and

$$|p(K) - p(K')| \leq e^{-rT} |K - K'|.$$



Theorem 9.19

c and p are **convex** in K , i.e.,

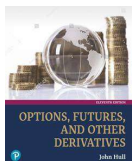
$$K \leq K' \text{ and } 0 < \alpha < 1$$

implies

$$c(\alpha K + (1 - \alpha)K') \leq \alpha c(K) + (1 - \alpha)c(K')$$

and

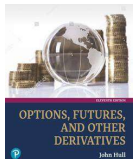
$$p(\alpha K + (1 - \alpha)K') \leq \alpha p(K) + (1 - \alpha)p(K').$$



Theorem 9.20

If $S \leq S'$, then

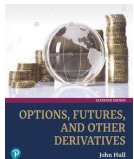
- $c(S) \leq c(S')$ and $p(S) \geq p(S')$
- $c(S') - c(S) \leq S' - S$
- $p(S) - p(S') \leq S' - S$
- c and p are convex in S .



Theorem 9.21

For American options, Proposition 9.17 holds, Theorem 9.18 (with $T=0$ there) holds, and Theorems 9.19 and 9.20 hold. Finally, if $T \leq T'$, then

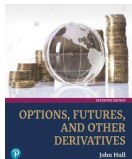
$$C(T) \leq C(T') \text{ and } P(T) \leq P(T').$$



Remark 9.22

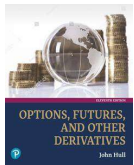
Summary of the effect on the stock option price when the parameter is increased while the other ones are kept constant:

	EC	EP	AC	AP
K	-	+	-	+
S	+	-	+	-
T	?	?	+	+
r	+	-	+	-
D	-	+	-	+
σ	+	+	+	+



Chapter 10

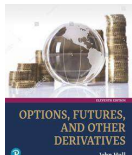
Trading Strategies involving Options



Definition 10.1

Positions in **an option and the underlying**:

- Writing a **covered call** (long stock, short call)
- Reverse of writing a covered call (short stock, long call)
- **Protective put** strategy (long stock, long put)
- Reverse of protective put strategy (short stock, short put)



Example 10.2

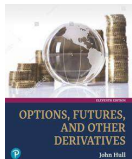
Recall: The relevant profit functions are:

- $S_T - S_0 e^{rT}$ **LONG STOCK**
- $(S_T - K)^+ - ce^{rT}$ **LONG CALL**
- $(K - S_T)^+ - pe^{rT}$ **LONG PUT**
- $S_0 e^{rT} - S_T$ **SHORT STOCK**
- $ce^{rT} - (S_T - K)^+$ **SHORT CALL**
- $pe^{rT} - (K - S_T)^+$ **SHORT PUT**

The profits for the strategies in Definition 10.1 are

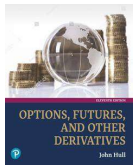
(for European options):

- $pe^{rT} - (K - S_T)^+$ **COVERED CALL**
- $(K - S_T)^+ - pe^{rT}$ **REVERSE CC**
- $(S_T - K)^+ - ce^{rT}$ **PROTECTIVE PUT**
- $ce^{rT} - (S_T - K)^+$ **REVERSE PP**



Definition 10.3

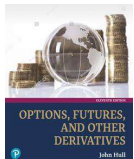
A **spread** is a position in two or more options of the same type.



Example 10.4

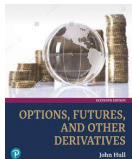
We discuss the following spreads:

- bull spread
- bear spread
- box spread
- butterfly spread
- calendar spread
- diagonal spread



Definition 10.5

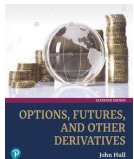
A **combination** is a position in both calls and puts on the same stock.



Example 10.6

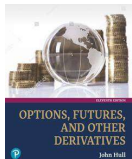
We discuss the following combinations:

- straddle
- strip
- strap
- strangle



Chapter 11

Binomial Trees

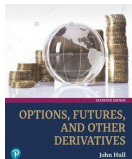


Remark 11.1

We let $T > 0$ and write $S_n = S(nT)$ for integers n . We assume S_{n+1} is either $S_n u$ with probability p or $S_n d$ with probability $1-p$, where

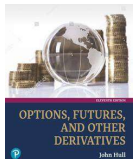
$$0 < d < 1 < u \text{ and } 0 < p < 1$$

(“stock price follows random walk”).



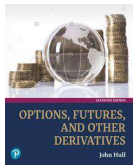
Example 11.2

Suppose $S_0=20$, $u=1.1$, $d=0.9$. Consider a European call on the stock with strike price $K=21$ and maturity $T=1/4$, $r=0.12$. We seek a risk-less portfolio in stock and option.



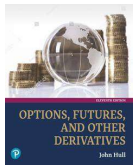
Definition 11.3

By a **European derivative security** or **contingent claim** with stock S as the underlying asset we mean a **random variable** of the form $f(S(T))$, where f is a given function, called the **payoff**.



Example 11.4

- call: $f(S) = (S - K)^+$
- put: $f(S) = (K - S)^+$
- long forward: $f(S) = S - K$

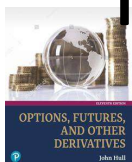


Theorem 11.5

In the one-step binomial tree model,

$$f = E_*(f(S_1))e^{-rT}$$

(“the present value of the contingent claim is equal to the discounted payoff expectation in a risk-neutral world, independent of p ”). Here $p_* = (e^{rT} - d) / (u - d)$.



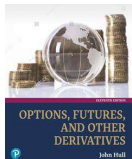
Example 11.6

Suppose $S_0=20$, $u=1.1$, $d=0.9$,
 $f_u=1$, $f_d=0$, $r=0.12$, $T=0.25$. In the
one-step BM, find c .

Theorem 11.7

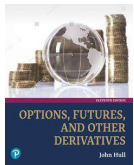
In the n-step binomial tree model,

$$f = E_*(f(S_n))e^{-rnT}.$$



Example 11.8

- Example 11.6 with $n=2$
- 2-year European put, $K=52$,
 $S_0=50$, $r=0.05$, $T=1$, $u=1.2$,
 $d=0.8$
- American put (as above)



Theorem 11.9

(Cox-Ross-Rubinstein formula)

In the binomial model, the price of a European call and put with strike price K to be exercised after N time steps is

$$c = S_0 [1 - \Phi(m-1, N, q)] - Ke^{-rNT} [1 - \Phi(m-1, N, p_*)]$$

Theorem 11.9 (continued)

and

$$p = -S_0 \Phi(m-1, N, q) + Ke^{-rNT} \Phi(m-1, N, p_*),$$

where Φ is the cdf of the binomial distribution, $q = p_* u e^{-rT}$, and m is the smallest nonnegative integer with

$$S_0 u^m d^{N-m} > K.$$

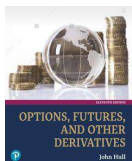
Remark 11.10

The **expected return** μ and the **volatility** σ of a stock price will be defined in such a way that

$$E(S_1) = S_0 e^{\mu T} \text{ and } \text{Var}(S_1) = S_0^2 \sigma^2 T$$

for small T in a one-step BM. Then

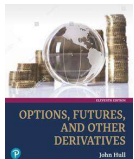
$$u = e^{\sigma T^{1/2}} \text{ and } d = e^{-\sigma T^{1/2}}$$



Example 11.11

- Example 11.8(c), where we assume that volatility is 30%

$$p = (a - d) / (u - d).$$

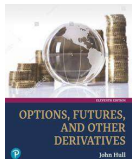


Example 11.11 (continued)

- For stocks paying a continuous dividend yield, use formulas as above except

$$a = e^{(r-q)T}.$$

E.g., stock index with $S_0=810$,
 $\sigma=0.2$, $q=0.02$, $r=0.05$, $K=800$,
 $T=0.25$, $N=2$, European call.

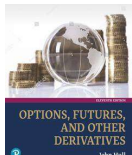


Example 11.11 (continued)

- Foreign currencies can be regarded as an asset providing a yield r_f , use

$$a = e^{(r-r_f)T}.$$

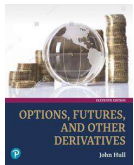
E.g., AUD is currently worth 0.61 USD and this exchange rate has a volatility of 12%, $r_f=0.07$, $r=0.05$. Value a 3-month American call with strike price of 0.6 using a three-step tree.



Example 11.11 (continued)

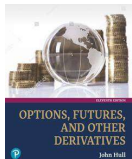
- Options on futures: Futures price should have an expected growth rate of zero. Use previous formulas with

$$a=1.$$



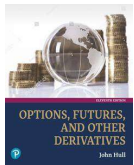
Chapter 12

Wiener Processes and Itô's Lemma



Definition 12.1

A **stochastic process** is a family of random variables $X=X(t)$, where t could be integers or real numbers.



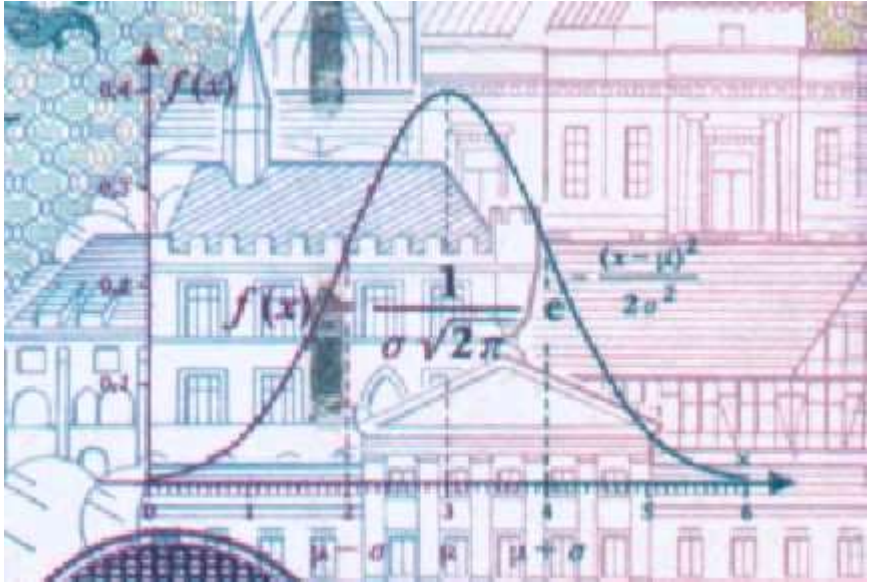
Remark 12.2 (Normal RV)

$X \sim N(0,1)$
 $P(X \leq 0) =$
 $P(X \leq 0.2) =$
 $P(X \leq 0.22) =$
 $P(X \leq -0.2) =$
 $P(X \leq x) = 0.95$

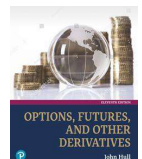
$X \sim N(2,4)$
 $P(X \leq 2.2) =$

$P(X \leq 1) =$

$P(X \leq x) = 0.95$



If $X \sim N(\mu, \sigma)$, then $(X - \mu) / \sigma \sim N(0,1)$



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	1	2	3	4	5	6	7	8	9	10	11	12
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4800	0.4760	0.4720	0.4680	0.4640	0.4600	0.4560
0.1	0.5398	0.5358	0.5318	0.5278	0.5238	0.5198	0.5158	0.5118	0.5078	0.5038	0.4998	0.4958
0.2	0.5793	0.5753	0.5713	0.5673	0.5633	0.5593	0.5553	0.5513	0.5473	0.5433	0.5393	0.5353
0.3	0.6179	0.6139	0.6099	0.6059	0.6019	0.5979	0.5939	0.5899	0.5859	0.5819	0.5779	0.5739
0.4	0.6554	0.6514	0.6474	0.6434	0.6394	0.6354	0.6314	0.6274	0.6234	0.6194	0.6154	0.6114
0.5	0.6915	0.6875	0.6835	0.6795	0.6755	0.6715	0.6675	0.6635	0.6595	0.6555	0.6515	0.6475
0.6	0.7264	0.7224	0.7184	0.7144	0.7104	0.7064	0.7024	0.6984	0.6944	0.6904	0.6864	0.6824
0.7	0.7602	0.7562	0.7522	0.7482	0.7442	0.7402	0.7362	0.7322	0.7282	0.7242	0.7202	0.7162
0.8	0.7924	0.7884	0.7844	0.7804	0.7764	0.7724	0.7684	0.7644	0.7604	0.7564	0.7524	0.7484
0.9	0.8224	0.8184	0.8144	0.8104	0.8064	0.8024	0.7984	0.7944	0.7904	0.7864	0.7824	0.7784
1.0	0.8500	0.8460	0.8420	0.8380	0.8340	0.8300	0.8260	0.8220	0.8180	0.8140	0.8100	0.8060

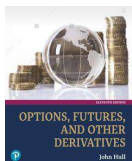
Definition 12.3

A stochastic process follows a **Wiener process** if

- the change ΔW during a small period of time Δt is

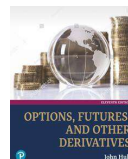
$$\Delta W = \varepsilon(\Delta t)^{1/2}, \text{ where } \varepsilon \sim N(0,1),$$

- the values of ΔW for any two different short intervals of time Δt are **independent**.



Example 12.4

Suppose W follows a Wiener process and time is measured in years. Suppose the value of W is initially 25. What is the value of W at the end of one year? What is the value of W at the end of five years? Find $P(W(1) > 26)$ and $P(W(5) > 26)$.



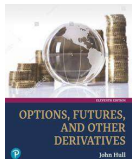
Year	Value of W
0	25
1	25.1
2	25.2
3	25.3
4	25.4
5	25.5

Definition 12.5

A stochastic process X follows a **generalized Wiener process** with drift rate a and variance rate b^2 if

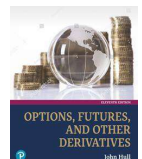
$$dX = a dt + b dW,$$

where W is a Wiener process.



Example 12.6

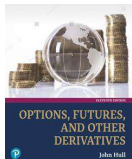
Assume that the cash position of a company, measured in thousands of dollars, follows a generalized Wiener process with a drift of 20 per year and a variance rate of 900 per year. Initially, the cash position is 50. Find the probabilities of negative cash positions after 1 year and after 3 months.



Year	1	2	3	4	5	6	7	8	9	10
Mean	50	70	90	110	130	150	170	190	210	230
Std Dev	30	42.43	54.74	67.05	79.36	91.67	103.98	116.29	128.60	140.91
Prob	0.0044	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Definition 12.7

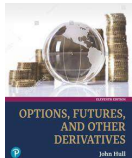
An **Itô process** is a generalized Wiener process in which the parameters **a** and **b** are functions of **X** and **t**.



Remark 12.8

Now we discuss the stochastic process usually assumed for the price S of a non-dividend-paying stock. A generalized Wiener process $dS=adt+bdW$ is not appropriate as

- expected percentage change of S should remain constant, not its expected absolute change,
- uncertainty as to the size of future stock price movements should be proportional to the level of the stock price.

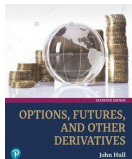


Remark 12.8 (continued)

So we are using the Itô process

$$dS = \mu S dt + \sigma S dW,$$

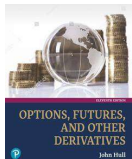
where μ is the **expected return** and σ is the **volatility** of the stock price. This model can be regarded as the limiting case of the random walk represented by binomial trees as the time step becomes smaller. The model is also known as **geometric Brownian motion**.



Example 12.10

Here we discuss **Monte-Carlo simulation**. Let $\mu=0.14$, $\sigma=0.2$, $\Delta t=0.01$.

t	S(t)	ϵ	$\Delta S(t)$
0	20.000	0.52	
0.01		1.44	
0.02		-0.86	
0.03		1.46	
0.04		-0.69	
0.05		-0.74	
0.06		0.21	
0.07		-1.1	
0.08		0.73	
0.09		1.16	
0.1		2.56	



Theorem 12.11

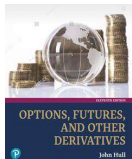
(Itô's lemma)

If dW is a WP,

$$dX = a(X, t)dt + b(X, t)dW,$$

and $G = G(x, t)$, then

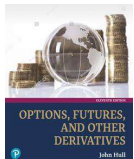
$$dG = (G_x a + G_t + G_{xx} b^2 / 2)dt + G_x b dW.$$



Corollary 12.12

If $dS = \mu S dt + \sigma S dW$ and $G = G(s, t)$,
then

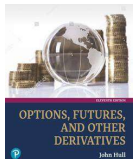
$$dG = (G_s \mu S + G_t + G_{ss} \sigma^2 S^2 / 2) dt + G_s \sigma S dW.$$



Corollary 12.13

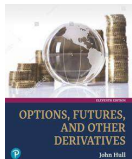
If $dS = \mu S dt + \sigma S dW$ and F is the forward price of a forward contract on the non-dividend-paying stock, then

$$dF = (\mu - r)F dt + \sigma F dW.$$



Remark 12.14

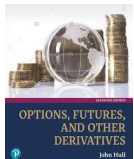
Note that F in Corollary 12.13 follows again geometric Brownian motion with the same variance rate as the stock price and a growth rate equal to the excess return of the stock price over the risk-free rate.



Corollary 12.15

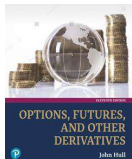
If $dS = \mu S dt + \sigma S dW$ and $G = \ln(S)$,
then

$$dG = (\mu - \sigma^2/2) dt + \sigma dW.$$



Chapter 13

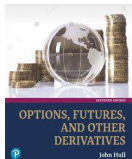
The Black-Scholes- Merton Model



Definition 13.1

A nonnegative random variable X is said to have a **lognormal distribution** with parameters μ and σ if

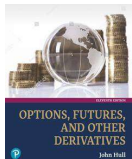
$$\ln(X) \sim N(\mu, \sigma).$$



Example 13.2

- If S is a stock price following geometric Brownian motion, then S_T has a lognormal distribution, namely

$$\ln(S_T) \sim N(\ln(S_0) + (\mu - \sigma^2/2)T, \sigma T^{1/2}).$$



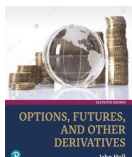
Theorem 13.3

If X has a lognormal distribution with parameters μ and σ , then

$$E(X) = e^{\mu + \sigma^2/2}$$

and

$$\text{Var}(X) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1).$$



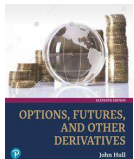
Example 13.4

- If S is a stock price following geometric Brownian motion, then

$$E(S_T) = S_0 e^{\mu T}$$

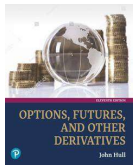
and

$$\text{Var}(S_T) = S_0^2 e^{2\mu T} (e^{\sigma^2 T} - 1).$$



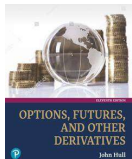
Example 13.4 (continued)

- Consider a stock with current price of \$20, an expected return of 20% (pa), and a volatility of 40% (pa). Find the expected value and the variance of the stock price in one year.



Remark 13.5

Here we discuss how to estimate volatility.

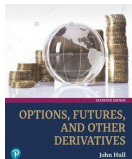


Theorem 13.6

(Black-Scholes-Merton PDE)

If $dS = \mu S dt + \sigma S dW$ and f is the price of a call, then

$$f_t + rSf_s + \sigma^2 S^2 f_{ss} / 2 = rf.$$



Remark 13.7

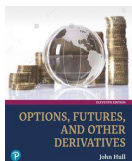
- (BSM) has many solutions, but we are looking for a solution that satisfies the boundary condition

$$f=(S-K)^+ \text{ when } t=T$$

for a European call, or

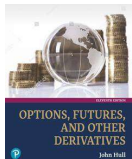
$$f=(K-S)^+ \text{ when } t=T$$

for a European put.



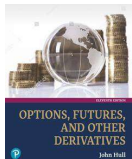
Remark 13.7 (continued)

- The portfolio from the proof of Theorem 13.6 is not permanently riskless, only during Δt . To keep the portfolio riskless, frequent adjustments are to be made.



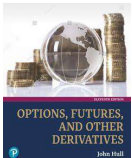
Remark 13.7 (continued)

- Any f that satisfies (BSM) is called a price of a **tradeable derivative**.



Example 13.8

- $f(S,t) = S - Ke^{-r(T-t)}$ is a price of a tradeable derivative.
- $f(S,t) = e^S$ is not a price of a tradeable derivative.
- $f(S,t) = e^{(\sigma^2 - 2r)(T-t)} / S$ is a price of a tradeable derivative.



Theorem 13.9

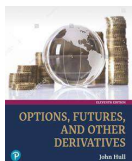
(Black-Scholes pricing formulas)

The prices at time 0 of a European call and put on a non-dividend-paying stock are

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

and

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1).$$



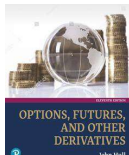
Theorem 13.9 (continued)

Here, N is the cdf of the standard normal distribution and

$$d_1 = (\ln(S_0/K) + (r + \sigma^2/2)T) / (\sigma T^{1/2})$$

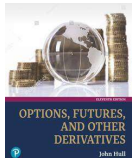
and

$$d_2 = (\ln(S_0/K) + (r - \sigma^2/2)T) / (\sigma T^{1/2}).$$



Remark 13.10

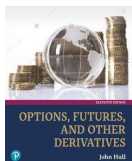
- Let $S_0 \rightarrow \infty$
- Let $\sigma \rightarrow 0$
- For N we can use the polynomial approximation providing 6-decimal-place accuracy $N(x) = 1 - N'(x)(a_1 k + a_2 k^2 + a_3 k^3 + a_4 k^4 + a_5 k^5)$ if $x \geq 0$ and $N(x) = 1 - N(-x)$ if $x < 0$. Here $k = 1/(1 + x\eta)$, $\eta = 0.2316419$, $a_1 = 0.319381530$, $a_2 = -0.356563782$, $a_3 = 1.781477937$, $a_4 = -1.821255978$, $a_5 = 1.330274429$.



Remark 13.12

We can also use risk-neutral valuation to prove BSPF:

- Assume that the expected return from the stock price is the risk-free rate.
- Calculate the expected payoff from the option.
- Discount at the risk-free rate.



Theorem 13.13

Let X be lognormally distributed with $\ln X \sim N(m, w)$. Then for $K > 0$,

$$E((X-K)^+) = E(X)N(d_1) - KN(d_2),$$

where

$$d_1 = (\ln(E(X)/K) + w^2/2)/w,$$

$$d_2 = (\ln(E(X)/K) - w^2/2)/w.$$

