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Chapter 11

Binomial Trees

Math 5730/Econ 5337, Fall 2025

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Remark 11.1

We let $T > 0$ and write $S_n = S(nT)$ for integers n . We assume S_{n+1} is either $S_n u$ with probability p or $S_n d$ with probability $1-p$, where $0 < d < 1 < u$ and $0 < p < 1$ ("stock price follows random walk").

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Example 11.2

Suppose $S_0 = 20$, $u = 1.1$, $d = 0.9$. Consider a European call on the stock with strike price $K = 21$ and maturity $T = 1/4$, $r = 0.12$. We seek a risk-less portfolio in stock and option.

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Definition 11.3

By a **European derivative security** or **contingent claim** with stock S as the underlying asset we mean a **random variable** of the form $f(S(T))$, where f is a given function, called the **payoff**.

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Example 11.4

- call: $f(S) = (S - K)^+$
- put: $f(S) = (K - S)^+$
- long forward: $f(S) = S - K$

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Theorem 11.5

In the one-step binomial tree model,

$$f = E \cdot (f(S_1)) e^{-rT}$$
 ("the present value of the contingent claim is equal to the discounted payoff expectation in a risk-neutral world, independent of p "). Here $p_* = (e^{rT} - d) / (u - d)$.

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Example 11.6

Suppose $S_0=20$, $u=1.1$, $d=0.9$, $K=21$, $r=0.12$, $T=0.25$.

- In the one-step BM, find c .
- Find also p .
- Show that PCP holds.

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Theorem 11.7

In the n -step binomial tree model,

$$f = E_+(f(S_n))e^{-rnT}.$$

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Example 11.8

- Example 11.6 with $n=2$
- 2-year European put, $K=52$, $S_0=50$, $r=0.05$, $T=1$, $u=1.2$, $d=0.8$
- American put (as above)

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Theorem 11.9

(Cox-Ross-Rubinstein formula)

In the binomial model, the price of a European call and put with strike price K to be exercised after N time steps is

$$c = S_0[1 - \Phi(m-1, N, q)] - Ke^{-rNT}[1 - \Phi(m-1, N, p_*)]$$

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Theorem 11.9 (continued)

and

$$p = -S_0\Phi(m-1, N, q) + Ke^{-rNT}\Phi(m-1, N, p_*),$$

where Φ is the cdf of the binomial distribution, $q = p_*ue^{-rT}$, and m is the smallest nonnegative integer with

$$S_0u^md^{N-m} > K.$$

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Remark 11.10

The **expected return** μ and the **volatility** σ of a stock price will be defined in such a way that

$$E(S_1) = S_0e^{\mu T} \text{ and } \text{Var}(S_1) = S_0^2\sigma^2T$$

for small T in a one-step BM. Then

$$u = e^{\sigma T^{1/2}} \text{ and } d = e^{-\sigma T^{1/2}}$$

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Example 11.11

- Example 11.8(c), where we assume that volatility is 30%

$$p=(a-d)/(u-d).$$

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Example 11.11 (continued)

- For stocks paying a continuous dividend yield, use formulas as above except

$$a=e^{(r-q)T}.$$

E.g., stock index with $S_0=810$, $\sigma=0.2$, $q=0.02$, $r=0.05$, $K=800$, $T=0.25$, $N=2$, European call.

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Example 11.11 (continued)

- Foreign currencies can be regarded as an asset providing a yield r_f , use

$$a=e^{(r-r_f)T}.$$

E.g., AUD is currently worth 0.61 USD and this exchange rate has a volatility of 12%, $r_f=0.07$, $r=0.05$. Value a 3-month American call with strike price of 0.6 using a three-step tree.

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Example 11.11 (continued)

- Options on futures: Futures price should have an expected growth rate of zero. Use previous formulas with

$$a=1.$$

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