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Chapter 13

The Black-Scholes-Merton Model

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Definition 13.1

A nonnegative random variable X is said to have a **lognormal distribution** with parameters μ and σ if

$$\ln(X) \sim N(\mu, \sigma).$$

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Example 13.2

- If S is a stock price following geometric Brownian motion, then S_T has a lognormal distribution, namely

$$\ln(S_T) \sim N(\ln(S_0) + (\mu - \sigma^2/2)T, \sigma T^{1/2}).$$

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Example 13.2 (continued)

- Consider a stock with an initial price of \$40, an expected return of 16% (pa), and a volatility of 20% (pa). Find a 95%-confidence interval for $S_{1/2}$.

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Theorem 13.3

If X has a lognormal distribution with parameters μ and σ , then

$$E(X) = e^{\mu + \sigma^2/2}$$

and

$$\text{Var}(X) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1).$$

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Example 13.4

- If S is a stock price following geometric Brownian motion, then

$$E(S_T) = S_0 e^{\mu T}$$

and

$$\text{Var}(S_T) = S_0^2 e^{2\mu T} (e^{\sigma^2 T} - 1).$$

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Example 13.4 (continued)

- Consider a stock with current price of \$20, an expected return of 20% (pa), and a volatility of 40% (pa). Find the expected value and the variance of the stock price in one year.

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Remark 13.5

Here we discuss how to estimate volatility.

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Theorem 13.6

(Black-Scholes-Merton PDE)

If $dS = \mu S dt + \sigma S dW$ and f is the price of a call, then

$$f_t + rSf_s + \sigma^2 S^2 f_{ss} / 2 = rf.$$

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Remark 13.7

- (BSM) has many solutions, but we are looking for a solution that satisfies the boundary condition

$f = (S - K)^+$ when $t = T$

for a European call, or

$f = (K - S)^+$ when $t = T$

for a European put.

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Remark 13.7 (continued)

- The portfolio from the proof of Theorem 13.6 is not permanently riskless, only during Δt . To keep the portfolio riskless, frequent adjustments are to be made.

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Remark 13.7 (continued)

- Any f that satisfies (BSM) is called a price of a **tradeable derivative**.

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Example 13.8

- $f(S,t)=S-Ke^{-r(T-t)}$ is a price of a tradeable derivative.
- $f(S,t)=e^S$ is not a price of a tradeable derivative.
- $f(S,t)=e^{(\sigma^2-2r)(T-t)}/S$ is a price of a tradeable derivative.

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Theorem 13.9

(Black-Scholes pricing formulas)

The prices at time 0 of a European call and put on a non-dividend-paying stock are

$$c = S_0 N(d_+) - Ke^{-rT} N(d_-)$$

and

$$p = Ke^{-rT} N(-d_-) - S_0 N(-d_+).$$

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Theorem 13.9 (continued)

Here, N is the cdf of the standard normal distribution and

$$d_+ = (\ln(S_0/K) + (r + \sigma^2/2)T) / (\sigma T^{1/2})$$

and

$$d_- = (\ln(S_0/K) + (r - \sigma^2/2)T) / (\sigma T^{1/2}).$$

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Remark 13.10

- Let $S_0 \rightarrow \infty$
- Let $\sigma \rightarrow 0$
- For N we can use the polynomial approximation providing 6-decimal-place accuracy $N(x) = 1 - N'(x)(a_1 k + a_2 k^2 + a_3 k^3 + a_4 k^4 + a_5 k^5)$ if $x \geq 0$ and $N(x) = 1 - N(-x)$ if $x < 0$. Here $k = 1/(1 + \eta)$, $\eta = 0.2316419$, $a_1 = 0.319381530$, $a_2 = 0.356563782$, $a_3 = 1.781477937$, $a_4 = 1.821255978$, $a_5 = 1.330274429$.

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Example 13.11

$T=1/2$, $S_0=42$, $K=40$, $r=0.1$, $\sigma=0.2$.
Find the prices of a European call and a European put.

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Remark 13.12

We can also use risk-neutral valuation to prove BSPF:

- Assume that the expected return from the stock price is the risk-free rate.
- Calculate the expected payoff from the option.
- Discount at the risk-free rate.

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Theorem 13.13

Let X be lognormally distributed with $\ln X \sim N(m, w)$. Then for $K > 0$,

$$E((X-K)^+) = E(X)N(d_+) - KN(d_-),$$

where

$$d_+ = (\ln(E(X)/K) + w^2/2)/w,$$

$$d_- = (\ln(E(X)/K) - w^2/2)/w.$$

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