Chapter 8
Risk-Neutral Valuation

8.1 Risk-Neutral Measure

Definition 8.1. Suppose $Z \geq 0$ is a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(Z) = 1$. Define a new probability measure $\tilde{\mathbb{P}}$ by the formula

$$\tilde{\mathbb{P}}(A) = \int_A Z \mathrm{d}\mathbb{P} \quad \text{for all} \quad A \in \mathcal{F}.$$

Then we define the *Radon–Nikodým derivative process* by

$$Z(t) = \mathbb{E}(Z|\mathcal{F}(t)) \quad \text{for all} \quad 0 \leq t \leq T.$$

Lemma 8.2. *If Y is $\mathcal{F}(t)$-measurable, then $\tilde{\mathbb{E}}(Y) = E(YZ(t))$.*

Lemma 8.3. *If $0 \leq s \leq t \leq T$ and Y is $\mathcal{F}(t)$-measurable, then*

$$\tilde{\mathbb{E}}(Y|\mathcal{F}(s)) = \frac{1}{Z(s)}\mathbb{E}(YZ(t)|\mathcal{F}(s)).$$

Theorem 8.4 (Girsanov). *Let $W(t)$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathbb{F}$ be a filtration for this Brownian motion. Let $\Theta(t)$ be an adapted process. Define*

$$Z(t) = \exp\left\{ -\int_0^t \Theta(u)\mathrm{d}W(u) - \frac{1}{2}\int_0^t \Theta^2(u)\mathrm{d}u \right\}$$

and

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)\mathrm{d}u,$$

where we assume

$$\mathbb{E}\left(\int_0^T \Theta^2(u)Z^2(u)\mathrm{d}u \right) < \infty.$$

Let $Z = Z(T)$. Then $\mathbb{E}(Z) = 1$, and under $\tilde{\mathbb{P}}$ defined by

$$\tilde{\mathbb{P}}(A) = \int_A Z \mathrm{d}\mathbb{P} \quad \textit{for all} \quad A \in \mathcal{F},$$

the process $\tilde{W}(t)$ is a Brownian motion.

Definition 8.5. Let R be an adapted *interest rate process*. We call D defined by

$$D(t) = \exp\left\{-\int_0^t R(s)\mathrm{d}s\right\}$$

the *discount process*. If S is any adapted process, then DS is called the *discounted process*. A measure $\tilde{\mathbb{P}} \sim \mathbb{P}$ is called *risk neutral* if under it the discounted stock price process is a martingale.

Theorem 8.6. *Suppose a stock price process follows generalized geometric Brownian motion with mean rate of return $\alpha(t)$ and volatility $\sigma(t)$. Define the* market price of risk

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}.$$

Then $\tilde{\mathbb{P}}$ defined in Girsanov's theorem is a risk-neutral measure, under which the stock price process follows again generalized geometric Brownian motion with volatility $\sigma(t)$ and mean rate of return $R(t)$.

Definition 8.7. A *portfolio process* is an adapted process. For a given initial wealth X_0 and a portfolio process φ, the *wealth process* X is defined to be the solution of

$$\mathrm{d}X(t) = \varphi(t)\mathrm{d}S(t) + R(t)\left(X(t) - \varphi(t)S(t)\right)\mathrm{d}t.$$

Theorem 8.8. *Under the risk-neutral measure defined in Girsanov's theorem, the discounted wealth process is a martingale.*

Remark 8.9. Under the risk-neutral measure defined in Girsanov's theorem, changes in the discounted wealth process of an agent's portfolio are entirely due to fluctuations in the discounted stock price.

Definition 8.10. An $\mathcal{F}(T)$-measurable random variable $V(T)$ is called a *European derivative security*. It is called *hedgeable* if there exists an initial capital $X(0)$ and a portfolio process φ such that the wealth process defined in Definition 8.7 satisfies $X(T) = V(T)$ almost surely.

Theorem 8.11. *If a European derivative security $V(T)$ is hedgeable, then, under the risk-neutral measure defined in Girsanov's theorem,*

$$X(t) = \tilde{\mathbb{E}}\left(e^{-\int_t^T R(u)\mathrm{d}u} V(T) | \mathcal{F}(t)\right) \quad \textit{for all} \quad 0 \le t \le T.$$

Theorem 8.12. *Let $V(T)$ be any European derivative security and define the* value process *by*

$$V(t) = \tilde{\mathbb{E}}\left(e^{-\int_t^T R(u)\mathrm{d}u} V(T) | \mathcal{F}(t)\right) \quad \textit{for all} \quad 0 \le t \le T.$$

Then the discounted value process is a martingale under the risk-neutral measure defined in Girsanov's theorem.

Theorem 8.13. *If $\sigma(t) \neq 0$ almost surely, then any European derivative security is hedgeable.*

Example 8.14. Assume that the interest rate R and the volatility σ are constant. Using the risk-neutral pricing formula, we can *derive* the price at time t of a European call with expiration time T and strike price K:

$$V(t) = S(t)N(d_+(T-t, S(t))) - Ke^{-r(T-t)}N(d_-(T-t, S(t))),$$

where

$$d_\pm(\tau, x) = \frac{\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

In particular,

$$V(0) = S(0)N(d_+(T, S(0))) - Ke^{-rT}N(d_-(T, S(0))).$$

8.2 Dividend-Paying Stocks

Definition 8.15. We say a stock pays *continuous dividends* over time at a rate $A(t)$ per unit time, where $A(t)$ is a nonnegative adapted process, if the stock price process and the wealth process are given by

$$\mathrm{d}S(t) = \alpha(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t) - A(t)S(t)\mathrm{d}t$$

and

$$\mathrm{d}X(t) = \varphi(t)\mathrm{d}S(t) + A(t)\varphi(t)S(t)\mathrm{d}t + R(t)(X(t) - \varphi(t)S(t))\mathrm{d}t.$$

Theorem 8.16. *If the stock pays continuous dividends at rate $A(t)$ per unit time, then the discounted wealth process is a martingale under the risk-neutral probability $\tilde{\mathbb{P}}$ given in Girsanov's theorem. While the discounted stock price process is not a martingale under $\tilde{\mathbb{P}}$, the interest-rate-discounted process at time t of an account that initially purchases one share of the stock and continuously reinvests the dividends in the stock, i.e.,*

$$\exp\left\{\int_0^t A(u)\mathrm{d}u\right\} D(t)S(t),$$

is a martingale under $\tilde{\mathbb{P}}$.

Theorem 8.17. *Let the volatility σ, the dividend rate a, and the interest rate r be constant. The price of a European call at time t, where the expiration time is T and the strike price is K, is given by*

$$V(t) = S(t)e^{-a(T-t)}N(d_+^a(T-t, S(t))) - Ke^{-r(T-t)}N(d_-^a(T-t, S(t))),$$

where

$$d_\pm^a(\tau, x) = \frac{\ln\left(\frac{x}{K}\right) + \left(r - a \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

In particular,

$$V(0) = S(0)e^{-aT}N(d_+^a(T, S(0))) - Ke^{-rT}N(d_-^a(T, S(0))).$$

Definition 8.18. We say a stock pays *lump payments of dividends* a_j at times $0 < t_1 < t_2 < \ldots < t_n < T$, where a_j are $\mathcal{F}(t_j)$-measurable for $1 \leq j \leq n$, if the stock price process and the wealth process are given by

$$\mathrm{d}S(t) = \alpha(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t), \quad t_j \leq t < t_{j+1},$$

$$S(t_{j+1}) = (1 - a_{j+1})S(t_{j+1}^-), \quad 0 \leq j \leq n,$$

where $t_0 := 0$, $t_{n+1} := T$, $a_0 := 0$, $a_{n+1} := 0$, and

$$\mathrm{d}X(t) = R(t)X(t)\mathrm{d}t + \varphi(t)\sigma(t)S(t)\mathrm{d}\tilde{W}(t), \quad 0 \leq t \leq T.$$

Remark 8.19. At dividend payment dates, value of portfolio stock holdings drop by $a_j\varphi(t_j)S(t_j^-)$, but the portfolio collects the dividend $a_j\varphi(t_j)S(t_j^-)$, so the portfolio value remains as above.

Theorem 8.20. *Let the volatility σ, the interest rate r, and the lump payments of dividends a_j be constant. The price of a European call at time 0, where the expiration time is T and the strike price is K, is given by*

$$V(0) = S(0)\left\{\prod_{j=1}^{n}(1 - a_j)\right\}N(d_+^{a_j}(T, S(0))) - Ke^{-rT}N(d_-^{a_j}(T, S(0))),$$

where

$$d_\pm^{a_j}(\tau, x) = \frac{\ln\left(\frac{x}{K}\right) + \sum_{j=1}^{n}\ln(1 - a_j) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

8.3 Forwards and Futures

Definition 8.21. A *zero-coupon bond* with face value K and expiration time T is a contract that pays K at time T.

Theorem 8.22. *The price at time t of a zero-coupon bond paying 1 at time T is*

$$B(t,T) = \frac{1}{D(t)} \tilde{\mathbb{E}}(D(T)|\mathcal{F}(t)).$$

Definition 8.23. A *forward contract* is an agreement to pay a specified delivery price K at a delivery time T for an asset whose price at time t is $S(t)$. The T*-forward price* $\text{For}_S(t,T)$ of this asset at time t is the value of K that makes the forward contract have price 0 at time t.

Theorem 8.24. *If zero-coupon bonds of all maturities can be traded, then*

$$\text{For}_S(t,T) = \frac{S(t)}{B(t,T)} \quad \textit{for all} \quad 0 \le t \le T.$$

Definition 8.25. The *futures price* of an asset whose value at time T is $S(T)$ is given by

$$\text{Fut}_S(t,T) = \tilde{\mathbb{E}}(S(T)|\mathcal{F}(t)).$$

Theorem 8.26. *The forward-futures spread is*

$$\text{For}_S(0,T) - \text{Fut}_S(0,T) = \frac{\mathbb{C}\text{ov}(D(T),S(T))}{B(0,T)}.$$

Theorem 8.27. *If the risk-free interest rate r is constant, then*

$$\text{For}_S(t,T) = \text{Fut}_S(t,T) = S(t)e^{r(T-t)}.$$

Theorem 8.28 (Black's Futures Option Formula). *If the volatility σ and the interest rate r are constant, then the price of a European futures call option is*

$$e^{-r(T-t)} \left\{ f(t)N(d_+^f(T-t,f(t))) - KN(d_-^f(T-t,f(t))) \right\},$$

where

$$f(t) = \text{Fut}_S(t,T^*) \quad \textit{and} \quad d_\pm^f(\tau,x) = \frac{\ln\left(\frac{x}{K}\right) \pm \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}}.$$