FIRST AND SECOND ORDER LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. We consider first and second order linear dynamic equations on a time scale. Such equations contain as special cases differential equations, difference equations, q-difference equations, and others. Important properties of the exponential function for a time scale are presented, and we use them to derive solutions of first and second order linear dyamic equations with constant coefficients. Wronskians are used to study equations with non-constant coefficients. We consider the reduction of order method as well as the method of variation of constants for the nonhomogeneous case. Finally, we use the exponential function to present solutions of the Euler-Cauchy dynamic equation on a time scale.

1. INTRODUCTION

The theory of measure chains, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis [14] in 1988 (supervised by Bernd Aulbach) in order to unify continuous and discrete analysis. Hilger [15] has defined an exponential function, trigonometric functions, and hyperbolic functions and developed some theory for linear dynamic equations with constant coefficients for time scales $\mathbb{T}$ (i.e., closed subsets of the reals) with constant graininess. The only time scales that

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have constant graininess are closed real intervals or sets which have constant step size $h > 0$. In this paper we are concerned with the general case when the time scale need not have constant graininess. The trigonometric and hyperbolic functions that we define are different from the ones defined by Hilger. Because of all the useful properties that we develop in this paper, we believe that this is the way these functions should be defined for the general setting.

Let us briefly summarize the set up of this paper. In the next Section 2 we give a short introduction to the time scales calculus, while the following Section 3 is devoted to the exponential function as introduced by Stefan Hilger in [16]. We also develop several important properties of the exponential function, which are needed in the remaining parts of this paper. Section 4 deals with first order linear dynamic equations on time scales. We present the solution of an initial value problem involving such (possibly nonhomogeneous) dynamic equations. Associated with a first order equation is a so-called adjoint equation, which is also of first order, and a study of this equation is contained in Section 4 as well. In Section 5 we discuss second order linear dynamic equations. We introduce Wronskians and derive Abel's formula. Next, we completely discuss the case of constant coefficients, and to do so, we introduce hyperbolic and trigonometric functions on time scales and derive some of their properties. Solutions of initial value problems involving an equation with constant coefficients are presented. Other special cases of second order equations (with not necessarily constant coefficients) are covered in Section 5 also, as well as a variation of constants result for the nonhomogeneous case. Finally, in Section 7, we discuss Euler-Cauchy equations on time scales, and present its solutions in terms of the exponential function.

2. The Time Scales Calculus

In this section we briefly introduce the time scales calculus. For proofs and further explanations and results we refer to the papers by Hilger [3, 15, 16], to the book by Kaymakçalan, Lakshmikantham, and Sivasundaram [17], and to the more recent papers [1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13]. A *time scale* $\mathbb{T}$ is a closed subset of $\mathbb{R}$, and the (forward) *jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$), while the *graininess* $\mu : \mathbb{T} \to \mathbb{R}_+$ is

$$\mu(t) := \sigma(t) - t.$$

A point t is called right-scattered if $\sigma(t) > t$ and right-dense if $\sigma(t) = t$. The notions of left-scattered and left-dense are defined similarly using the backward jump operator. We write $\mathbb{T}^\kappa$ for $\mathbb{T}$ minus a possible left-scattered maximum. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if the left-hand sided limit exists at left-dense points. For a function $f : \mathbb{T} \to \mathbb{R}$ we define the derivative f^Δ as follows: Let $t \in \mathbb{T}$. If there exists a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U,$$

then f is said to be (delta) *differentiable* at t, and we call α the derivative of f at t and denote it by $f^\Delta(t)$. Moreover, we denote $f^\sigma = f \circ \sigma$. The following formulas are useful:

- $f^\sigma = f + \mu f^\Delta$;
- $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta$ ("Product Rule");
- $(f/g)^\Delta = (f^\Delta g - f g^\Delta)/(g g^\sigma)$ ("Quotient Rule").

A function F with $F^\Delta = f$ is called an *antiderivative* of f, and then we define

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

where $a, b \in \mathbb{T}$. It is well-known [15] that rd-continuous functions possess antiderivatives. A simple consequence of the first formula above is

- $\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)$.

3. The Exponential Function

A function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive* if

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

Hilger [16] showed that for $t_0 \in \mathbb{T}$ and rd-continuous and regressive p, the solution of the initial value problem

$$y^\Delta = p(t)y, \quad y(t_0) = 1 \tag{3.1}$$

is given by $e_p(\cdot, t_0)$, where

$$e_p(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \quad \text{with} \quad \xi_h(z) = \begin{cases} \frac{\operatorname{Log}(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0. \end{cases}$$

Clearly, $e_p(t,s)$ never vanishes, and hence we can consider the quotient $f/e_p(\cdot, t_0)$ if f is another solution of (3.1). But then

$$\begin{aligned} \left(\frac{f}{e_p(\cdot,t_0)}\right)^\Delta &= \frac{f^\Delta e_p(\cdot,t_0) - f e_p^\Delta(\cdot,t_0)}{e_p(\cdot,t_0)e_p^\sigma(\cdot,t_0)} \\ &= \frac{pfe_p(\cdot,t_0) - fpe_p(\cdot,t_0)}{e_p(\cdot,t_0)e_p^\sigma(\cdot,t_0)} \\ &= 0 \end{aligned}$$

so $f/e_p(\cdot, t_0) \equiv f(t_0)/e_p(t_0, t_0) = 1$ and hence $e_p(\cdot, t_0)$ is the only solution of (3.1).

We now proceed to give some fundamental properties of the exponenetial function. To do so it is necessary to introduce the following notation: For regressive $p, q : \mathbb{T} \to \mathbb{R}$ we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := -\frac{p}{1+\mu p}, \quad p \ominus q := p \oplus (\ominus p).$$

Theorem 3.1. *Assume $p, q : \mathbb{T} \to \mathbb{R}$ are regressive and rd-continuous, then the following hold:*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$.

Proof. Part (i). The function $y(t) \equiv 1$ is obviously a solution of the initial value problem $y^\Delta = 0y$, $y(s) = 1$, and since this problem has only one solution, namely $e_0(t,s)$, we have that $e_0(t,s) \equiv y(t) \equiv 1$.

Part (ii). By a formula from Section 2 we have

$$\begin{aligned} e_p(\sigma(t),s) &= e_p^\sigma(t,s) \\ &= e_p(t,s) + \mu(t)e_p^\Delta(t,s) \\ &= (1+\mu(t)p(t))e_p(t,s). \end{aligned}$$

Part (iii). Consider the initial value problem

$$y^\Delta = (\ominus p)(t)y, \quad y(s) = 1. \tag{3.2}$$

It is easy to see that the dynamic equation in (3.2) is regressive. We show that for each fixed s, $y(t) = 1/e_p(t,s)$ satisfies (3.2). Indeed, $y(s) = 1$ is obvious. We use the quotient rule to obtain

$$\begin{aligned} y^\Delta(t) &= \left(\frac{1}{e_p(\cdot,s)}\right)^\Delta(t) \\ &= -\frac{e_p^\Delta(t,s)}{e_p(t,s)e_p(\sigma(t),s)} \\ &= -\frac{p(t)}{e_p(\sigma(t),s)} \\ &= -\frac{p(t)}{(1+\mu(t)p(t))e_p(t,s)} \\ &= (\ominus p)(t)y(t), \end{aligned}$$

where we have also used part (ii) in the second to the last step.

Part (iv). By the definition of the exponential function, it follows that

$$e_p(t,s) = \frac{1}{e_p(s,t)},$$

and this is equal to $e_{\ominus p}(s,t)$ according to part (iii).

Part (v). Consider the initial value problem

$$y^\Delta = p(t)y, \quad y(r) = 1. \tag{3.3}$$

We show that $y(t) = e_p(t,s)e_p(s,r)$ satisfies (3.3): It is obvious that $y^\Delta(t) = p(t)y(t)$, and $y(r) = e_p(r,s)e_p(s,r) = 1$ follows from part (iv).

Part (vi). Consider the initial value problem

$$y^\Delta = (p \oplus q)(t)y, \quad y(s) = 1. \tag{3.4}$$

It is easy to see that the dynamic equation in (3.4) is regressive. We show that $y(t) = e_p(t,s)e_q(t,s)$ satisfies (3.4): We have $y(s) = e_p(s,s)e_q(s,s) = 1$, and we use the product rule to calculate

$$\begin{aligned} y^{\Delta}(t) &= (e_p(\cdot,s)e_q(\cdot,s))^{\Delta}(t) \\ &= p(t)e_p(t,s)e_q(\sigma(t),s) + e_p(t,s)q(t)e_q(t,s) \\ &= p(t)e_p(t,s)(1+\mu(t)q(t))e_q(t,s) + e_p(t,s)q(t)e_q(t,s) \\ &= (p \oplus q)(t)y(t), \end{aligned}$$

where we have also used part (ii) of this theorem.

Part (vii). This follows easily using parts (iii) and (vi) of this theorem. □

Theorem 3.2. *Assume that $p : \mathbb{T} \to \mathbb{R}$ is regressive and rd-continuous. Let $t_0 \in \mathbb{T}$.*

(i) *If $1 + \mu p > 0$ on $\mathbb{T}$, then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$;*

(ii) *If $1 + \mu p < 0$ on $\mathbb{T}^{\kappa}$, then $e_p(t, t_0) = \alpha(t, t_0)(-1)^{n_t}$ for all $t \in \mathbb{T}$, where*

$$\alpha(t,t_0) := \exp\left(\int_{t_0}^{t} \frac{\log|1+\mu(\tau)p(\tau)|}{\mu(\tau)}\Delta\tau\right) > 0$$

and n_t is one plus the (finite) number of elements of $\mathbb{T}$ strictly between t_0 and t.

Proof. Part (i): Since $1 + \mu(t)p(t) > 0$, we have $\mathrm{Log}[1 + \mu(t)p(t)] \in \mathbb{R}$ for all $t \in \mathbb{T}$ and therefore

$$\xi_{\mu(t)}(p(t)) \in \mathbb{R} \quad \text{for all} \quad t \in \mathbb{T}.$$

Hence $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$, by the definition of the exponential function.

Part (ii) follows similarly as above: Since $\mu(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$, we have

$$\mathrm{Log}[1+\mu(t)p(t)] = \log|1+\mu(t)p(t)| + i\pi \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}$$

and therefore

$$\begin{aligned} e_p(t,t_0) &= \exp\left(\int_{t_0}^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\ &= \exp\left(\int_{t_0}^t \frac{\operatorname{Log}(1+\mu(\tau)p(\tau))}{\mu(\tau)}\Delta\tau\right) \\ &= \exp\left(\int_{t_0}^t \frac{\log|1+\mu(\tau)p(\tau)|+i\pi}{\mu(\tau)}\Delta\tau\right) \\ &= \exp\left(\int_{t_0}^t \left\{\frac{\log|1+\mu(\tau)p(\tau)|}{\mu(\tau)}+\frac{i\pi}{\mu(\tau)}\right\}\Delta\tau\right) \\ &= \alpha(t,t_0)\exp\left(i\pi\int_{t_0}^t \frac{\Delta\tau}{\mu(\tau)}\right) \\ &= \alpha(t,t_0)\exp\left(i\pi n_t\right) \\ &= \alpha(t,t_0)(-1)^{n_t}, \end{aligned}$$

where we used a formula from Section 2 to evaluate the last integral. □

Example 3.1. (i) If $\mathbb{T}=\mathbb{R}$, then $e_p(t,s)=\exp\left\{\int_s^t p(\tau)\Delta\tau\right\}$ for continuous p, hence $e_\alpha(t,s)=e^{\alpha(t-s)}$ for constant α, and $e_1(t,0)=e^t$.

(ii) If $\mathbb{T}=\mathbb{Z}$, then $e_p(t,s)=\prod_{\tau=s}^{t-1}(1+p(\tau))$ if p is never -1 (and for $s<t$), hence $e_\alpha(t,s)=(1+\alpha)^{t-s}$ for $\alpha\neq-1$, and $e_1(t,0)=2^t$.

(iii) If $\mathbb{T}=h\mathbb{Z}=\{hk|\ k\in\mathbb{Z}\}$ for $h>0$, then $e_1(t,0)=(1+h)^{t/h}$, in particular if $h=\frac{1}{n}$ for $n\in\mathbb{N}$, then $e_1(t,0)=\left[\left(1+\frac{1}{n}\right)^n\right]^t$.

(iv) If $\mathbb{T}=q^{\mathrm{N}_0}=\left\{q^k|\ k\in\mathbb{N}_0\right\}$ where $q>1$ (this case corresponds to so-called q-difference equations, see e.g., [4, 18, 19]), then it is easy to verify that $y(t)=\sqrt{t}\exp(-\ln^2(t)/(2\ln(q)))$ is equal to $e_p(t,1)$, if

$p(t) := (1-t)/((q-1)t^2)$. To see this note that $y(1) = 1$ and

$$\begin{aligned} y^\Delta(t) &= \frac{y(\sigma(t)) - y(t)}{\mu(t)} \\ &= \frac{\sqrt{qt}\exp\left(-\frac{(\ln(q)+\ln(t))^2}{2\ln(q)}\right) - \sqrt{t}\exp\left(-\frac{\ln^2(t)}{2\ln(q)}\right)}{(q-1)t} \\ &= \frac{\left[\sqrt{q}\exp\left(-\frac{\ln(q)}{2}\right)\exp(-\ln(t)) - 1\right]\sqrt{t}\exp\left(-\frac{\ln^2(t)}{2\ln(q)}\right)}{(q-1)t} \\ &= \frac{1-t}{(q-1)t^2}y(t) \\ &= p(t)y(t). \end{aligned}$$

(v) If $\mathbb{T} = \mathbb{N}_0^2 = \{k^2 \mid k \in \mathbb{N}_0\}$, then $y(t) = 2^{\sqrt{t}}(\sqrt{t})!$ is equal to $e_1(t,0)$. To see this note that $y(0) = 1$ and

$$\begin{aligned} y^\Delta(t) &= \frac{y(\sigma(t)) - y(t)}{\mu(t)} \\ &= \frac{2^{\sqrt{t}+1}(\sqrt{t}+1)! - 2^{\sqrt{t}}(\sqrt{t})!}{2\sqrt{t}+1} \\ &= y(t). \end{aligned}$$

(vi) If $\mathbb{T} = \left\{\sum_{k=1}^{n} \frac{1}{k} \mid k \in \mathbb{N}\right\}$, then $e_\alpha(t,s) = \binom{n-s+\alpha}{n-s}$, where $t = \sum_{k=1}^{n} \frac{1}{k}$.

4. First Order Linear Dynamic Equations

In this section we present solutions to initial value problems involving first order linear nonhomogeneous dynamic equations with rd-continuous coefficients.

Theorem 4.1. *Let $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous and regressive. Suppose $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, $t_0 \in \mathbb{T}$, and $x_0 \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$x^\Delta = -p(t)x^\sigma + f(t), \quad x(t_0) = x_0 \tag{4.1}$$

is given by

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau.$$

Proof. Using the properties of the exponential function given in Theorem 3.1 it is easy to verify that x given above indeed solves (4.1). Conversely, let us assume that x is a solution of (4.1). We multiply both sides of the dynamic equation in (4.1) by the integrating factor $e_p(t, t_0)$ and obtain

$$\begin{aligned} [e_p(\cdot, t_0)x]^{\Delta} &= e_p(\cdot, t_0)x^{\Delta} + pe_p(\cdot, t_0)x^{\sigma} \\ &= e_p(\cdot, t_0)\left[x^{\Delta} + px^{\sigma}\right] \\ &= e_p(\cdot, t_0)f, \end{aligned}$$

and now we integrate both sides from t_0 to t to conclude

$$e_p(t, t_0)x(t) - e_p(t_0, t_0)x(t_0) = \int_{t_0}^{t} e_p(\tau, t_0)f(\tau)\Delta\tau. \tag{4.2}$$

This integration is possible according to Section 2 since f is rd-continuous. Solving equation (4.2) for $x(t)$ leads to the desired result. □

Theorem 4.2. *Let $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous and regressive. Suppose $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, $t_0 \in \mathbb{T}$, and $y_0 \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$y^{\Delta} = p(t)y + f(t), \quad y(t_0) = y_0 \tag{4.3}$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Proof. We equivalently rewrite $y^{\Delta} = py + f$ as $y^{\Delta} = p(y^{\sigma} - \mu y^{\Delta}) + f$, i.e.,

$$(1 + \mu p)y^{\Delta} = py^{\sigma} + f,$$

i.e., $y^{\Delta} = \frac{p}{1+\mu p}y^{\sigma} + \frac{f}{1+\mu p}$, i.e.,

$$y^{\Delta} = -(\ominus p)y^{\sigma} + \frac{f}{1 + \mu p}.$$

By Theorem 4.1, the unique solution of (4.3) is hence given by (note that $\ominus(\ominus p) = p$)

$$e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \tau)\frac{f(\tau)}{1 + \mu(\tau)p(\tau)}\Delta\tau.$$

Using (ii) from Theorem 3.1, we obtain the desired result. □

5. Second Order Linear Dynamic Equations

The *Wronskian* of two differentiable functions x_1 and x_2 is defined by

$$W(x_1, x_2) = \det \begin{pmatrix} x_1 & x_2 \\ x_1^\Delta & x_2^\Delta \end{pmatrix}.$$

We first consider the equation

$$x^{\Delta\Delta} + p(t)x^{\Delta^\sigma} + q(t)x^\sigma = 0. \tag{5.1}$$

Theorem 5.1 (Abel's Formula)**.** *Let $p, q : \mathbb{T} \to \mathbb{R}$ and $t_0 \in \mathbb{T}$. Suppose that p is regressive. If x_1 and x_2 solve* (5.1)*, then*

$$W(x_1, x_2)(t) = W(x_1, x_2)(t_0)e_{\ominus p}(t, t_0) \quad \textit{for all} \quad t \in \mathbb{T}.$$

Proof. A simple calculation shows that

$$W^\Delta(x_1, x_2) = \det \begin{pmatrix} x_1^\sigma & x_2^\sigma \\ x_1^{\Delta\Delta} & x_2^{\Delta\Delta} \end{pmatrix} = -pW^\sigma(x_1, x_2),$$

and hence our claim follows from Theorem 4.2. □

Corollary 5.1. *If $q : \mathbb{T} \to \mathbb{R}$, the Wronskian of any two solutions of*

$$x^{\Delta\Delta} + q(t)x^\sigma = 0$$

is constant.

Now we will find the solutions of the second order linear dynamic equation with constant coefficients

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0 \tag{5.2}$$

(where α and β are constant).

Theorem 5.2. *Suppose $\alpha^2 - 4\beta \neq 0$. If $\mu\beta - \alpha$ is regressive, then a fundamental system of* (5.2) *is given by*

$$e_{\lambda_1}(\cdot, t_0) \quad \textit{and} \quad e_{\lambda_2}(\cdot, t_0),$$

where $t_0 \in \mathbb{T}$ and

$$\lambda_1 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \quad \textit{and} \quad \lambda_2 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}.$$

The solution of the initial value problem

$$(5.2), \quad y(t_0) = y_0, \quad y^{\Delta}(t_0) = y_0^{\Delta} \tag{5.3}$$

is given by

$$y_0 \frac{e_{\lambda_1}(\cdot, t_0) + e_{\lambda_2}(\cdot, t_0)}{2} + \frac{\alpha y_0 + 2y_0^{\Delta}}{\sqrt{\alpha^2 - 4\beta}} \frac{e_{\lambda_1}(\cdot, t_0) - e_{\lambda_2}(\cdot, t_0)}{2}.$$

Proof. Since λ_1 and λ_2 given above solve of the *characteristic equation*

$$\lambda^2 + \alpha\lambda + \beta = 0, \tag{5.4}$$

we find that both $e_{\lambda_1}(\cdot, t_0)$ and $e_{\lambda_2}(\cdot, t_0)$ solve the dynamic equation (5.2). Moreover, the Wronskian of these two solutions is found to be

$$\begin{aligned} \det \begin{pmatrix} e_{\lambda_1}(t, t_0) & e_{\lambda_2}(t, t_0) \\ \lambda_1 e_{\lambda_1}(t, t_0) & \lambda_2 e_{\lambda_2}(t, t_0) \end{pmatrix} &= (\lambda_2 - \lambda_1) e_{\lambda_1}(t, t_0) e_{\lambda_2}(t, t_0) \\ &= \sqrt{\alpha^2 - 4\beta}\, e_{\lambda_1 \oplus \lambda_2}(t, t_0), \end{aligned}$$

which does not vanish unless $\alpha^2 - 4\beta = 0$. The solution of the initial value problem (5.3) is readily verified to be of the form as given above. ☐

In the next three theorems we find solutions of (5.2) involving hyperbolic functions if $\alpha^2 - 4\beta > 0$, trigonometric functions if $\alpha^2 - 4\beta < 0$, and we use Theorem 5.1 (i.e., the reduction of order method) to find solutions if $\alpha^2 - 4\beta = 0$.

Definition 5.1 (Hyperbolic Functions)**.** If $-\mu p^2$ is regressive, we define the hyperbolic functions

$$\cosh_p = \frac{e_p + e_{-p}}{2} \quad \text{and} \quad \sinh_p = \frac{e_p - e_{-p}}{2}.$$

In the following, if f is a function in two variables, we mean by f^Δ the derivative with respect to the first variable.

Lemma 5.1. *If $-\mu p^2$ is regressive, then we have*

$$\cosh_p^\Delta = p \sinh_p \quad \textit{and} \quad \sinh_p^\Delta = p \cosh_p$$

and

$$\cosh_p^2 - \sinh_p^2 = e_{-\mu p^2}.$$

Proof. Using Definition 5.1, the first two formulas are easily verified, while the last formula follows from

$$\begin{aligned} \cosh_p^2 - \sinh_p^2 &= \left(\frac{e_p + e_{-p}}{2}\right)^2 - \left(\frac{e_p - e_{-p}}{2}\right)^2 \\ &= e_p e_{-p} \\ &= e_{-p \oplus p} \\ &= e_{-\mu p^2}, \end{aligned}$$

where we have used Theorem 3.1 (vi). □

Theorem 5.3. *Suppose $\alpha^2 - 4\beta > 0$. Define*

$$p = -\frac{\alpha}{2} \quad \textit{and} \quad q = \frac{\sqrt{\alpha^2 - 4\beta}}{2}.$$

If p and $\mu\beta - \alpha$ are regressive, then a fundamental system of (5.2) *is given by*

$$\cosh_{q/(1+\mu p)}(\cdot, t_0) e_p(\cdot, t_0) \quad \textit{and} \quad \sinh_{q/(1+\mu p)}(\cdot, t_0) e_p(\cdot, t_0),$$

where $t_0 \in \mathbb{T}$, and the Wronskian of these two solutions is

$$q e_{\mu\beta - \alpha}(\cdot, t_0).$$

The solution of the initial value problem (5.3) *is given by*

$$\left[y_0 \cosh_{q/(1+\mu p)}(\cdot, t_0) + \frac{y_0^\Delta - p y_0}{q} \sinh_{q/(1+\mu p)}(\cdot, t_0) \right] e_p(\cdot, t_0).$$

Proof. In this proof we use the convention that

$$e_p = e_p(\cdot, t_0)$$

and similarly for cosh and sinh. We apply Theorem 5.2 to find two solutions of (5.2) as

$$e_{p+q} \quad \text{and} \quad e_{p-q}.$$

Therefore, we can construct two other solutions of (5.2) by

$$y_1 = \frac{e_{p+q} + e_{p-q}}{2} \quad \text{and} \quad y_2 = \frac{e_{p+q} - e_{p-q}}{2}.$$

We use the formulas

$$p \oplus \left(\frac{q}{1 + \mu p} \right) = p + \frac{q}{1 + \mu p} + \frac{\mu p q}{1 + \mu p} = p + q$$

and

$$p \oplus \left(-\frac{q}{1 + \mu p} \right) = p - \frac{q}{1 + \mu p} - \frac{\mu p q}{1 + \mu p} = p - q$$

to obtain, by using Theorem 3.1 (vi)

$$\begin{aligned} y_1 &= \frac{e_{p+q} + e_{p-q}}{2} \\ &= \frac{e_{p\oplus(q/(1+\mu p))} + e_{p\oplus(-q/(1+\mu p))}}{2} \\ &= \frac{e_p e_{q/(1+\mu p)} + e_p e_{-q/(1+\mu p)}}{2} \\ &= e_p \frac{e_{q/(1+\mu p)} + e_{-q/(1+\mu p)}}{2} \\ &= e_p \cosh_{q/(1+\mu p)} \end{aligned}$$

and similarly

$$y_2 = \frac{e_{p+q} - e_{p-q}}{2} = e_p \sinh_{q/(1+\mu p)} .$$

Next, we find by using Theorem 3.1 (ii) and Lemma 5.1

$$\begin{aligned} y_1^\Delta &= e_p^\Delta \cosh_{q/(1+\mu p)} + e_p^\sigma \cosh_{q/(1+\mu p)}^\Delta \\ &= p e_p \cosh_{q/(1+\mu p)} + e_p^\sigma \frac{q}{1+\mu p} \sinh_{q/(1+\mu p)} \\ &= p e_p \cosh_{q/(1+\mu p)} + q e_p \sinh_{q/(1+\mu p)} \end{aligned}$$

and similarly

$$y_2^\Delta = p e_p \sinh_{q/(1+\mu p)} + q e_p \cosh_{q/(1+\mu p)} .$$

Use

$$y_1(t_0) = 1, \quad y_2(t_0) = 0, \quad y_1^\Delta(t_0) = p, \quad y_2^\Delta(t_0) = q$$

to verify that the above given function indeed solves (5.3). Finally, we find the Wronskian of y_1 and y_2 as

$$\begin{aligned} W(y_1,y_2) &= \det\begin{pmatrix} y_1 & y_2 \\ y_1^\Delta & y_2^\Delta \end{pmatrix} \\ &= \det\begin{pmatrix} e_p\cosh_{q/(1+\mu p)} & e_p\sinh_{q/(1+\mu p)} \\ qe_p\sinh_{q/(1+\mu p)} & qe_p\cosh_{q/(1+\mu p)} \end{pmatrix} \\ &= qe_p^2\left[\cosh^2_{q/(1+\mu p)} - \sinh^2_{q/(1+\mu p)}\right] \\ &= qe_p^2 e_{-\mu q^2/(1+\mu p)^2} \\ &= qe_{p(2+\mu p)\oplus(-\mu q^2/(1+\mu p)^2)} \\ &= qe_{2p+\mu(p^2-q^2)} \\ &= qe_{\mu\beta-\alpha}, \end{aligned}$$

where we have used Lemma 5.1. □

Definition 5.2 (Trigonometric Functions)**.** If μp^2 is regressive, we define the trigonometric functions

$$\cos_p = \frac{e_{ip}+e_{-ip}}{2} \quad \text{and} \quad \sin_p = \frac{e_{ip}-e_{-ip}}{2i}.$$

The proofs of the following Lemma 5.2 and Theorem 5.4 are similar to the proofs of Lemma 5.1 and Theorem 5.3 and hence will be omitted.

Lemma 5.2. *If μp^2 is regressive, then we have*

$$\cos_p^\Delta = -p\sin_p \quad \textit{and} \quad \sin_p^\Delta = p\cos_p$$

and

$$\cos_p^2 + \sin_p^2 = e_{\mu p^2}.$$

Theorem 5.4. *Suppose $\alpha^2 - 4\beta < 0$. Define*

$$p = -\frac{\alpha}{2} \quad \textit{and} \quad q = \frac{\sqrt{4\beta-\alpha^2}}{2}.$$

If $p, q \in \mathbb{R}$ and p is regressive, then a fundamental system of (5.2) *is given by*

$$\cos_{q/(1+\mu p)}(\cdot, t_0)e_p(\cdot, t_0) \quad \text{and} \quad \sin_{q/(1+\mu p)}(\cdot, t_0)e_p(\cdot, t_0),$$

where $t_0 \in \mathbb{T}$, and the Wronskian of these two solutions is

$$qe_{\mu\beta-\alpha}(\cdot, t_0).$$

The solution of the initial value problem (5.3) *is given by*

$$\left[y_0 \cos_{q/(1+\mu p)}(\cdot, t_0) + \frac{y_0^\Delta - py_0}{q} \sin_{q/(1+\mu p)}(\cdot, t_0)\right] e_p(\cdot, t_0).$$

Finally, we treat the case that the characteristic equation (5.4) has a double zero. In this case Theorem 5.2 only gives one solution, while we can obtain a second linearly independent solution by using Theorem 5.1.

Theorem 5.5. *Suppose $\alpha^2 - 4\beta = 0$. Define*

$$p = -\frac{\alpha}{2}.$$

If p is regressive, then a fundametal system of (5.2) *is given by*

$$e_p(\cdot, t_0) \quad \text{and} \quad e_p(\cdot, t_0)\int_{t_0}^{t} \frac{1}{1 + p\mu(\tau)}\Delta\tau,$$

where $t_0 \in \mathbb{T}$, and the Wronskian of these two solutions is

$$e_{\mu p^2}.$$

The solution of the initial value problem (5.3) *is given by*

$$e_p(t, t_0)\left[y_0 + (y_0^\Delta - py_0)\int_{t_0}^{t} \frac{\Delta\tau}{1 + p\mu(\tau)}\right].$$

Proof. It is easy to directly verify all the statements given above. However, here we show how we obtain the second solution given above using Theorem 5.1. Clearly,

$$y_1 = e_p(\cdot, t_0)$$

is a solution of (5.2). Now let y_2 be another solution satisfying the initial conditions $y_2(t_0) = 0$ and $y_2^\Delta(t_0) = 1$ and consider

$$\begin{aligned} W(e_p(\cdot, t_0), y_2) &= e_p(\cdot, t_0) y_2^\Delta - e_p^\Delta(\cdot, t_0) y_2 \\ &= e_p(\cdot, t_0) y_2^\Delta - p e_p(\cdot, t_0) y_2 \\ &= (y_2^\Delta - p y_2) e_p(\cdot, t_0). \end{aligned}$$

On the other hand, by Abel's Theorem (Theorem 5.1), we have

$$W(e_p(\cdot, t_0), y_2)(t) = W(e_p(\cdot, t_0), y_2)(t_0) e_{\mu\beta-\alpha}(t, t_0) = e_{\mu\beta-\alpha}(t, t_0).$$

Hence y_2 is a solution of the first order linear equation

$$(y^\Delta - py) e_p(t, t_0) = e_{\mu\beta-\alpha}(t, t_0)$$

or, equivalently according to Theorem 3.1 (vii)

$$y^\Delta - py = e_{(\mu\beta-\alpha)\ominus p}(t, t_0) = e_p(t, t_0).$$

Since $y_2(t_0) = 0$ we have by the variation of constants formula given in Theorem 4.2 that

$$\begin{aligned} y_2(t) &= \int_{t_0}^t e_p(t, \sigma(\tau)) e_p(\tau, t_0) \Delta\tau \\ &= e_p(t, t_0) \int_{t_0}^t \frac{1}{1 + \mu(\tau) p} \Delta\tau, \end{aligned}$$

where we have used Theorem 3.1 again. □

Finally we will consider the nonhomogeneous equation

$$y^{\Delta\Delta} + p(t) y^\Delta + q(t) y = f(t), \tag{5.5}$$

where p and q need not be constant and f is an rd-continuous function. We remark that a reduction of order technique has been applied in order to find the solution given in Theorem 5.6 below. However, once the solutions are obtained, it is easy to verify that they are indeed solutions, and hence we will omit the proof of Theorem 5.6.

Theorem 5.6 (Variation of Parameters)**.** *Suppose that y_1 and y_2 form a fundamental set of solutions of the homogeneous equation* (5.2)*. Then the solution of the initial value problem*

$$(5.5), \quad y(t_0) = y_0, \quad y^\Delta(t_0) = y_0^\Delta$$

is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \int_{t_0}^{t} \frac{y_2(t) y_1(\sigma(\tau)) - y_1(t) y_2(\sigma(\tau))}{W(y_1, y_2)(\sigma(\tau))} f(\tau)\Delta\tau,$$

where the constants c_1 and c_2 are given by

$$c_1 = \frac{y_2^\Delta(t_0) y_0 - y_2(t_0) y_0^\Delta}{W(y_1, y_2)(t_0)}$$

and

$$c_2 = \frac{y_1(t_0) y_0^\Delta - y_1^\Delta(t_0) y_0}{W(y_1, y_2)(t_0)}.$$

6. Nonconstant coefficients

In general, there is no method to solve second order dynamic equations with arbitrary nonconstant coefficients. However, if one solution is known, it is possible to use the reduction of order method as discussed in the proof of Theorem 5.5 to find a second linearly independent solution. We illustrate this procedure with the dynamic equation (5.1).

For rd-continuous p we put

$$p^{②} = (-p) \cdot (\ominus p).$$

Theorem 6.1. *Suppose that z is regressive and solves*

$$z^\Delta + z^{②} + p(t) z^\sigma + q(t) = 0. \tag{6.1}$$

Then $e_z(\cdot, t_0)$ solves (5.1).

Proof. Suppose z solves (6.1) and let $x = e_z = e_z(\cdot, t_0)$. Then

$$x^\Delta = ze_z \quad \text{and hence} \quad (x^\Delta)^\sigma = z^\sigma e_z^\sigma$$

and

$$\begin{aligned} x^{\Delta\Delta} &= z^\Delta e_z^\sigma + z e_z^\Delta \\ &= z^\Delta e_z^\sigma + z^2 e_z \\ &= z^\Delta e_z^\sigma + \frac{z^2}{1+\mu z} e_z^\sigma \\ &= z^\Delta e_z^\sigma + z^{②} e_z^\sigma \\ &= \left(z^\Delta + z^{②}\right) e_z^\sigma. \end{aligned}$$

Altogether we have

$$\begin{aligned} x^{\Delta\Delta} + p(x^\Delta)^\sigma + qx^\sigma &= \left(z^\Delta + z^{②}\right) e_z^\sigma + pz^\sigma e_z^\sigma + qe_z^\sigma \\ &= \left(z^\Delta + z^{②} + pz^\sigma + q\right) e_z^\sigma \\ &= 0 \end{aligned}$$

since z solves (6.1). □

We now will find a second (linearly independent) solution x_2 of (5.1). Let x_2 be the solution of (5.1) satisfying

$$x_2(t_0) = 0 \quad \text{and} \quad x_2^\Delta(t_0) = 1.$$

Using Theorem 5.1 we obtain

$$\begin{aligned} e_z(\cdot, t_0)\left[x_2^\Delta - zx_2\right] &= W(x_1, x_2) \\ &= e_{\ominus p}(\cdot, t_0) W(x_1, x_2)(t_0) \\ &= e_{\ominus p}(\cdot, t_0). \end{aligned}$$

Therefore, by the quotient rule from Section 2,

$$\begin{aligned}\left(\frac{x_2}{e_z(\cdot,t_0)}\right)^{\Delta} &= \frac{x_2^{\Delta}e_z(\cdot,t_0)-x_2e_z^{\Delta}(\cdot,t_0)}{e_z(\cdot,t_0)e_z^{\sigma}(\cdot,t_0)}\\ &= \frac{e_{\ominus p}(\cdot,t_0)}{e_z(\cdot,t_0)e_z^{\sigma}(\cdot,t_0)}\end{aligned}$$

and hence

$$\begin{aligned}x_2(t) &= e_z(t,t_0)\int_{t_0}^{t}\frac{e_{\ominus p}(\tau,t_0)}{e_z(\tau,t_0)e_z^{\sigma}(\tau,t_0)}\Delta\tau\\ &= e_z(t,t_0)\int_{t_0}^{t}\frac{e_{z\oplus z\oplus p}(t_0,\tau)}{1+\mu(\tau)z(\tau)}\Delta\tau.\end{aligned}$$

Hence it is easy to verify the following result.

Theorem 6.2. *If z solves equation* (6.1), *then*

$$e_z(t,t_0)\left[x_0+(x_0^{\Delta}-x_0z(t_0))\int_{t_0}^{t}\frac{e_{z\oplus z\oplus p}(t_0,\tau)}{1+\mu(\tau)z(\tau)}\Delta\tau\right]$$

is the solution of the initial value problem

$$(5.1),\quad x(t_0)=x_0,\quad x^{\Delta}(t_0)=x_0^{\Delta}.$$

In order to apply Theorem 6.1, it is crucial to find a solution of equation (6.1). As is checked readily, this is an easy task if (5.1) is of the form (5.2). Another example, in which (6.1) can be solved explicitly, is given next.

Example 6.1. Let q be constant and regressive and consider the equation

$$x^{\Delta\Delta}-q^{②}(t)x^{\sigma}=0. \tag{6.2}$$

Then a solution of equation (6.1) is given by

$$z=q.$$

By Theorem 6.1,

$$e_q(t,t_0)\quad\text{and}\quad e_q(t,t_0)\int_{t_0}^{t}\frac{e_q^2(t_0,\tau)}{[1+q\mu(\tau)]}\Delta\tau$$

form a fundamental system of (6.2). The solution of the initial value problem

$$(6.2), \quad x(t_0) = x_0, \quad x^\Delta(t_0) = x_0^\Delta$$

is given by

$$e_q(t,t_0)\left[x_0 + (x_0^\Delta - qx_0)\int_{t_0}^t \frac{e_q^2(t_0,\tau)}{1+q\mu(\tau)}\Delta\tau\right].$$

Finally we consider an equation given in self-adjoint form

$$(p(t)x^\Delta)^\Delta + q(t)x^\sigma = 0. \tag{6.3}$$

The p given below in Theorem 6.3 is not necessarily a differentiable function (unless the graininess of the time scale is constant. Hence (6.3) can not be rewritten in the form (5.1) or (5.2).

Theorem 6.3. *Suppose α is regressive. Let $\tilde{p}$ be rd-continuous and nonvanishing. We put*

$$p = \frac{\tilde{p}}{\ominus\alpha} \quad \textit{and} \quad q = \alpha\tilde{p} - \tilde{p}^\Delta.$$

Then a fundamental system for (6.3) is given by

$$e_{\ominus\alpha}(t,t_0) \quad \textit{and} \quad -p(t_0)\int_{t_0}^t \frac{\alpha(\tau)}{\tilde{p}(\tau)} e_\alpha(\tau,t)e_\alpha(\tau,t_0)\Delta\tau.$$

The Wronskian of these two solutions is

$$p(t_0).$$

Proof. All the claims above are easily verified. □

Example 6.2. Suppose α is regressive. We let $\tilde{p} = e_\alpha(\cdot,t_0)$ so that $q = 0$ in the above Theorem 6.3. Hence we consider the equation

$$\left(\frac{e_\alpha^\sigma(\cdot,t_0)}{\alpha}x^\Delta\right)^\Delta = 0. \tag{6.4}$$

By Theorem 6.3, a fundamental system for (6.4) is given by

$$e_{\ominus\alpha}(\cdot, t_0) \quad \text{and} \quad -\frac{1}{(\ominus\alpha)(t_0)}\left(1 - e_{\ominus\alpha}(\cdot, t_0)\right).$$

7. Euler-Cauchy Dynamic Equations

In this section we consider the *Euler-Cauchy dynamic equation*

$$t\sigma(t)y^{\Delta\Delta} + aty^{\Delta} + by = 0 \quad \text{with} \quad a, b \in \mathbb{R}. \tag{7.1}$$

We will only solve the equation (7.1) for $t \in \mathbb{T}$, $t > 0$. We assume that the regressivity condition

$$1 - \frac{a\mu(t)}{\sigma(t)} + \frac{b\mu^2(t)}{t\sigma(t)} \neq 0 \tag{7.2}$$

for $t \in \mathbb{T}$, $t > 0$ is satisfied. The associated *characteristic equation* of (7.1) is defined by

$$\lambda^2 + (a-1)\lambda + b = 0. \tag{7.3}$$

Theorem 7.1. *If the regressivity condition* (7.2) *is satisfied and the characteristic equation* (7.3) *has two distinct roots* λ_1 *and* λ_2*, then a fundamental system of* (7.1) *is given by*

$$e_{\lambda_1/t}(\cdot, t_0) \quad \textit{and} \quad e_{\lambda_2/t}(\cdot, t_0),$$

where $t_0 \in \mathbb{T}$, $t_0 > 0$. *If, in addition,* $1 + \mu(t)\frac{\lambda_i}{t} > 0$, $i = 1, 2$, $t \in \mathbb{T}$, $t > 0$, *then the above exponential functions form a fundamental set of positive solutions of the Euler-Cauchy dynamic equation* (7.1) *on* $\mathbb{T}$, $t > 0$.

Proof. Let $t_0 \in \mathbb{T}$, $t_0 > 0$ and let

$$y(t) = e_{\lambda/t}(t, t_0)$$

where we assume that $1 + \mu(t)\frac{\lambda}{t} \neq 0$ for $t \in \mathbb{T}$, $t > 0$ so that the above exponential function exists. Then we have

$$y^{\Delta}(t) = \frac{\lambda}{t}y(t) \quad \text{and hence} \quad ty^{\Delta}(t) = \lambda y(t).$$

Furthermore we find

$$\begin{aligned} y^{\Delta\Delta}(t) &= -\frac{\lambda}{t\sigma(t)}y(t) + \frac{\lambda}{\sigma(t)}y^{\Delta}(t) \\ &= -\frac{\lambda}{t\sigma(t)}y(t) + \frac{\lambda}{\sigma(t)}\frac{\lambda}{t}y(t) \\ &= \frac{\lambda^2-\lambda}{t\sigma(t)}y(t) \end{aligned}$$

and hence

$$t\sigma(t)y^{\Delta\Delta}(t) = (\lambda^2-\lambda)y(t).$$

Therefore we have that

$$\begin{aligned} t\sigma(t)y^{\Delta\Delta}(t) + aty^{\Delta}(t) + by(t) &= (\lambda^2-\lambda)y(t) + a\lambda y(t) + by(t) \\ &= (\lambda^2 - \lambda + a\lambda + b)y(t) \\ &= \left(\lambda^2 + (a-1)\lambda + b\right)y(t). \end{aligned}$$

Now assume that λ_1 and λ_2 are distinct roots of the characteristic equation (7.3). It follows that

$$\lambda_1 + \lambda_2 = 1 - a \quad \text{and} \quad \lambda_1\lambda_2 = b.$$

Then (7.2) implies

$$\begin{aligned} \left(1+\mu(t)\frac{\lambda_1}{t}\right)\left(1+\mu(t)\frac{\lambda_2}{t}\right) &= \frac{1}{t^2}\left(t^2 + (\lambda_1+\lambda_2)t\mu(t) + \mu^2(t)\lambda_1\lambda_2\right) \\ &= \frac{1}{t^2}\left(t^2 + (1-a)t\mu(t) + b\mu^2(t)\right) \\ &= \frac{1}{t^2}\left(t\sigma(t) - at\mu(t) + b\mu^2(t)\right) \\ &= \frac{\sigma(t)}{t}\left(1 - \frac{a\mu(t)}{\sigma(t)} + \frac{b\mu^2(t)}{t\sigma(t)}\right) \end{aligned}$$

does not vanish for $t \in \mathbb{T}$, $t > 0$. Hence the exponential functions

$$e_{\lambda_1/t}(\cdot, t_0) \quad \text{and} \quad e_{\lambda_2/t}(\cdot, t_0)$$

are well-defined solutions of the Euler-Cauchy dynamic equation (7.1) on $\mathbb{T}$, $t > 0$. Note that

$$\begin{aligned} W\left[e_{\lambda_1}(\cdot, t_0), e_{\lambda_2}(\cdot, t_0)\right](t) &= \det \begin{pmatrix} e_{\frac{\lambda_1}{t}}(t, t_0) & e_{\frac{\lambda_2}{t}}(t, t_0) \\ \frac{\lambda_1}{t} e_{\frac{\lambda_1}{t}}(t, t_0) & \frac{\lambda_2}{t} e_{\frac{\lambda_2}{t}}(t, t_0) \end{pmatrix} \\ &= \frac{1}{t}(\lambda_2 - \lambda_1) e_{\frac{\lambda_1}{t} \oplus \frac{\lambda_2}{t}}(t, t_0) \\ &\neq 0 \end{aligned}$$

for $t \in \mathbb{T}$, $t > 0$ since $\lambda_1 \neq \lambda_2$. Hence the exponential functions

$$e_{\lambda_1/t}(\cdot, t_0) \quad \text{and} \quad e_{\lambda_2/t}(\cdot, t_0)$$

form a fundamental system of solutions of the Euler-Cauchy dynamic equation (7.1) on $\mathbb{T}$, $t > 0$. If the characteristic values λ_1 and λ_2 are such that $1 + \mu(t)\frac{\lambda_i}{t} > 0$, $i = 1, 2$, $t \in \mathbb{T}$, $t > 0$, then the exponential functions

$$e_{\lambda_1/t}(\cdot, t_0) \quad \text{and} \quad e_{\lambda_2/t}(\cdot, t_0)$$

form a fundamental set of positive solutions of the Euler-Cauchy dynamic equation (7.1) on $\mathbb{T}$, $t > 0$. □

Next we consider the Euler-Cauchy dynamic equation in the double root case.

Theorem 7.2. *Assume that* $\alpha \in \mathbb{R}$ *and* $t_0 \in \mathbb{T}$ *with* $t_0 > 0$. *If the regressivity condition*

$$1 - \frac{1 - 2\alpha}{\sigma(t)}\mu(t) + \frac{\alpha^2}{t\sigma(t)}\mu^2(t) \neq 0 \tag{7.4}$$

holds for $t \in \mathbb{T}$, $t > 0$, *then a fundamental system of the Euler-Cauchy dynamic equation*

$$t\sigma(t)y^{\Delta\Delta} + (1 - 2\alpha)ty^{\Delta} + \alpha^2 y = 0 \tag{7.5}$$

is given by

$$e_{\frac{\alpha}{t}}(t,t_0) \quad \textit{and} \quad e_{\frac{\alpha}{t}}(t,t_0)\int_{t_0}^{t}\frac{1}{\tau+\alpha\mu(\tau)}\Delta\tau$$

for $t\in\mathbb{T}$, $t>0$.

Proof. First note that

$$\begin{aligned}\left(1+\mu(t)\frac{\alpha}{t}\right)\left(1+\mu(t)\frac{(\alpha-1)}{\sigma(t)}\right) &= 1+\left[\frac{\alpha}{t}+\frac{(\alpha-1)}{\sigma(t)}\right]\mu(t)+\frac{\alpha(\alpha-1)}{t\sigma(t)}\mu^2(t)\\ &= 1+\left[\frac{\alpha\sigma(t)+(\alpha-1)t}{t\sigma(t)}\right]\mu(t)+\frac{\alpha(\alpha-1)}{t\sigma(t)}\mu^2(t)\\ &= 1+\left[\frac{\alpha\mu(t)+2\alpha t-t}{t\sigma(t)}\right]\mu(t)+\frac{\alpha(\alpha-1)}{t\sigma(t)}\mu^2(t)\\ &= 1-\frac{1-2\alpha}{\sigma(t)}\mu(t)+\frac{\alpha^2}{t\sigma(t)}\neq 0\end{aligned}$$

for $t\in\mathbb{T}$. This implies that the exponentials

$$e_{\frac{\alpha}{t}}(\cdot,t_0) \quad \text{and} \quad e_{\frac{\alpha-1}{\sigma(t)}}(\cdot,t_0)$$

are well-defined. The chacteristic equation of (7.5) is

$$\lambda^2-2\alpha+\alpha^2=0$$

and so the characteristic roots are $\lambda_1=\lambda_2=\alpha$. Hence one linearly independent solution of (7.5) is

$$y_1=e_{\frac{\alpha}{t}}(\cdot,t_0).$$

We will now use the method of reduction of order to show how to get a second linearly independent solution. First note that equation (7.5) can be written in the form

$$y^{\Delta\Delta}+p(t)y^{\Delta}+q(t)y=0$$

where

$$p(t)=\frac{1-2\alpha}{\sigma(t)} \quad \text{and} \quad q(t)=\frac{\alpha^2}{t\sigma(t)}.$$

It follows that

$$\begin{aligned}
-p(t)+\mu(t)q(t) &= \frac{2\alpha-1}{\sigma(t)}+\mu(t)\frac{\alpha^2}{t\sigma(t)} \\
&= \frac{2\alpha-1}{\sigma(t)}+[\sigma(t)-t]\frac{\alpha^2}{t\sigma(t)} \\
&= \frac{2\alpha-1}{\sigma(t)}+\frac{\alpha^2}{t}-\frac{\alpha^2}{\sigma(t)} \\
&= \frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}.
\end{aligned}$$

Now let $y_2(t)$ be a solution of (7.5) such that

$$W(e_{\frac{\alpha}{t}}(\cdot,t_0),y_2)=e_{-p+\mu q}(t,t_0)=e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}}(t,t_0)$$

which we can do by Abel's Theorem (Theorem 5.1). It follows that

$$\begin{aligned}
\frac{W(e_{\frac{\alpha}{t}}(\cdot,t_0),y_2)}{e_{\frac{\alpha}{t}}(t,t_0)e^{\sigma}_{\frac{\alpha}{t}}(t,t_0)} &= \frac{e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}}(t,t_0)}{e_{\frac{\alpha}{t}}(t,t_0)e^{\sigma}_{\frac{\alpha}{t}}(t,t_0)} \\
&= \frac{e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}}(t,t_0)}{[1+\mu(t)\frac{\alpha}{t}]e^2_{\frac{\alpha}{t}}(t,t_0)} \\
&= -\frac{t_0}{t+\alpha\mu(t)}.
\end{aligned}$$

By the quotient rule we get that y_2 satisfies the dynamic equation

$$\left(\frac{y_2(t)}{e_{\frac{\alpha}{t}}(t,t_0)}\right)^{\Delta}=-\frac{t_0}{t+\alpha\mu(t)}.$$

Integrating both sides from t_0 to t and assuming $y_2(t_0)=0$ we have

$$\frac{y_2(t)}{e_{\frac{\alpha}{t}}(t,t_0)}=-t_0\int_{t_0}^{t}\frac{1}{\tau+\alpha\mu(\tau)}\Delta\tau.$$

It follows that

$$y_2(t)=-t_0e_{\frac{\alpha}{t}}(t,t_0)\int_{t_0}^{t}\frac{1}{\tau+\alpha\mu(\tau)}\Delta\tau$$

is a solution of (7.5), and since

$$W(e_{\frac{\alpha}{t}}(\cdot,t_0),y_2)\neq 0,$$

we have that

$$e_{\frac{\alpha}{t}}(\cdot, t_0) \quad \text{and} \quad y_2$$

form a fundamental set of solutions of (7.5). But this implies that

$$e_{\frac{\alpha}{t}}(t, t_0) \quad \text{and} \quad e_{\frac{\alpha}{t}}(t, t_0) \int_{t_0}^{t} \frac{1}{\tau + \alpha\mu(\tau)} \Delta\tau$$

form a fundamental set of solutions of (7.5). □

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