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# THE STOCHASTIC DYNAMIC EXPONENTIAL AND GEOMETRIC BROWNIAN MOTION ON ISOLATED TIME SCALES

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**Abstract**

The mathematics of time scales has recently received much attention and holds great promise in a number of areas. In this paper we propose a new area of mathematics, namely the theory of stochastic dynamic equations, which unifies the theories of stochastic differential and difference equations. We give an example involving stochastic dynamic equations, namely an equation modeling a stock price.

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## 1 Introduction

The study of time scales in mathematics is an attempt to bridge the divide between the discrete and the continuous. Introduced by Hilger [3], time scales provides a unified framework for both difference equations and differential equations, and are the subject of the two books by Bohner and Peterson [1, 2]. In this paper we present an application of time scales in the field of stochastic dynamic equations with applications to stochastic volatility models [6].

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The organization of this paper is as follows. Section 2 presents a few core definitions and concepts of time scales. In Section 3 we define the stochastic integral and in Section 4 we introduce the concept of the stochastic exponential and study its basic properties. Finally in Section 5 we study geometric Brownian motion on time scales, and an example related to stock price is given in Section 6. Our attempt is to make the mathematical discussion that follows as self contained as is practical.

## 2 Preliminaries, Notation, and Assumptions

An arbitrary nonempty closed subset of the set of the real numbers $\mathbb{T}$ is referred to as a *time scale* and denoted by $\mathbb{T}$. The most common time scales are $\mathbb{T} = \mathbb{R}$ for continuous calculus, $\mathbb{T} = h\mathbb{Z} = \{hn : n \in \mathbb{Z}\}$ with $h > 0$ for discrete calculus, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$ with $q > 1$ for quantum calculus. We assume that $\mathbb{T}$ is unbounded above and define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{for all } t \in \mathbb{T}.$$

The *graininess* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t \quad \text{for all } t \in \mathbb{T}.$$

In this paper we assume that $\mathbb{T}$ is an *isolated time scale*, i.e., we have $\mu(t) > 0$ for all $t \in \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$, we define the *Hilger derivative* $f^\Delta$ by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \quad \text{for all } t \in \mathbb{T}. \tag{2.1}$$

Throughout we fix $t_0 \in \mathbb{T}$, let

$$\mathbb{T}_t = [t_0, t) \cap \mathbb{T} \quad \text{for all } t \in \mathbb{T}$$

and define the *Hilger integral* by

$$\int_{t_0}^t f(\tau)\Delta\tau = \sum_{\tau \in \mathbb{T}_t} \mu(\tau) f(\tau) \quad \text{for all } t \in \mathbb{T}. \tag{2.2}$$

Finally we say that $f$ is *regressive* and write $f \in \mathcal{R}$ provided

$$f(t) \neq -\frac{1}{\mu(t)} \quad \text{for all } t \in \mathbb{T}.$$

The following result is known from [1, Theorem 2.35].

**Theorem 2.1.** *If $p \in \mathcal{R}$, then the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1, \quad t \in \mathbb{T} \tag{2.3}$$

*has a unique solution.*

The unique solution of (2.3) is denoted by $e_p(\cdot, t_0)$ and called the *exponential function*.

## 3 Stochastic Integral

Given a time scale $\mathbb{T}$, a collection of measurable real functions $X = \{X(t) : t \in \mathbb{T}\}$, defined on a measurable space $(\Omega, \mathcal{F})$, will be referred to as a *stochastic process* indexed by $\mathbb{T}$. For a fixed point $\omega \in \Omega$, the function on $\mathbb{T}$ given by $t \mapsto X(t;\omega)$ is the sample path of the process $X$ corresponding to $\omega$. A filtration on $(\Omega, \mathcal{F})$ indexed by $\mathbb{T}$ is a family $\{\mathcal{F}(t) : t \in \mathbb{T}\}$ of sub-sigma algebras of $\mathcal{F}$ with the property that $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ for all $s$ and $t$ on $\mathbb{T}$ with $s < t$. If $X(t)$ is $\mathcal{F}(t)$-measurable for each $t \in \mathbb{T}$, then we shall refer to $\{X(t), \mathcal{F}(t) : t \in \mathbb{T}\}$ as an adapted stochastic process. Given a stochastic process $X$ indexed by $\mathbb{T}$, the simplest choice of a filtration is the one generated by the process itself, $\{\mathcal{F}^X(t) : t \in \mathbb{T}\}$, where $\mathcal{F}^X(t)$ denotes the smallest sigma algebra with respect to which $X(s)$ is measurable for all $s \in \mathbb{T}$ satisfying $s \le t$.

**Definition 3.1.** A Brownian motion indexed by a time scale $\mathbb{T}$ is an adapted stochastic process $W = \{W(t), \mathcal{F}(t) : t \in \mathbb{T}\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the following properties:

1. $W(t_0) = 0$ a.s.,
2. if $t_0 \le s < t$ and $s, t \in \mathbb{T}$, then the increment $W(t) - W(s)$ is independent of $\mathcal{F}(s)$ and is normally distributed with mean zero and variance $t - s$.

By Definition 3.1, we have that $\Delta W(t) := W(\sigma(t)) - W(t)$ is normally distributed with

$$\mathbb{E}[\Delta W(t)] = 0 \quad \text{and} \quad \mathbb{V}[\Delta W(t)] = \mu(t) \quad \text{for all } t \in \mathbb{T}. \tag{3.1}$$

**Definition 3.2.** For an adapted stochastic process $X$ and a Brownian motion $W$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we introduce the *stochastic dynamic integral*

$$\int_{t_0}^{t} X(\tau)\Delta W(\tau) = \sum_{\tau \in \mathbb{T}_t} X(\tau)\Delta W(\tau) \quad \text{for all } t \in \mathbb{T}. \tag{3.2}$$

**Definition 3.3.** Let $X_0$ be a random variable and $X_1$, $X_2$ be stochastic processes (indexed by $\mathbb{T}$) such that $X_2$ is adapted. Then

$$X(t) = X_0 + \int_{t_0}^{t} X_1(\tau)\Delta\tau + \int_{t_0}^{t} X_2(\tau)\Delta W(\tau) \quad \text{for all } t \in \mathbb{T}, \tag{3.3}$$

is called an *Itô process*, and instead of (3.3) we also write

$$\Delta X = X_1 \Delta t + X_2 \Delta W.$$

**Lemma 3.4.** *If* $\Delta X = X_1 \Delta t + X_2 \Delta W$*, then*

$$X(\sigma(t)) = X(t) + \mu(t)X_1(t) + X_2(t)\Delta W(t). \tag{3.4}$$

*Proof.* Since $\Delta X = X_1 \Delta t + X_2 \Delta W$, we have (3.3), and thus by (2.2) and (3.2),

$$\begin{aligned}
X(\sigma(t)) - X(t) = X_0 + \int_{t_0}^{\sigma(t)} X_1(\tau)\Delta\tau + \int_{t_0}^{\sigma(t)} X_2(\tau)\Delta W(\tau) \\
- X_0 - \int_{t_0}^{t} X_1(\tau)\Delta\tau - \int_{t_0}^{t} X_2(\tau)\Delta W(\tau) \\
= \sum_{\tau\in\mathbb{T}_{\sigma(t)}} X_1(\tau)\mu(\tau) + \sum_{\tau\in\mathbb{T}_{\sigma(t)}} X_2(\tau)\Delta W(\tau) - \sum_{\tau\in\mathbb{T}_t} X_1(\tau)\mu(\tau) - \sum_{\tau\in\mathbb{T}_t} X_1(\tau)\Delta W(\tau) \\
= X_1(t)\mu(t) + X_2(t)\Delta W(t)
\end{aligned}$$

as $\mathbb{T}_{\sigma(t)} \setminus \mathbb{T}_t = \{t\}$. This yields (3.4). □

**Lemma 3.5.** *If $\Delta X = X_1 \Delta t + X_2 \Delta W$ and $a : \mathbb{T} \to \mathbb{R}$, then*

$$\int_{t_0}^{t} a^{\Delta}(\tau)X(\tau)\Delta\tau + \int_{t_0}^{t} a(\sigma(\tau))X_1(\tau)\Delta\tau + \int_{t_0}^{t} a(\sigma(\tau))X_2(\tau)\Delta W(\tau) = a(t)X(t) - a(t_0)X(t_0). \tag{3.5}$$

*Proof.* Using Lemma 3.4, (2.1), (2.2) and (3.2), we obtain

$$\begin{aligned}
a(t)X(t) - a(t_0)X(t_0) &= \sum_{\tau\in\mathbb{T}_t} [a(\sigma(\tau))X(\sigma(\tau)) - a(\tau)X(\tau)] \\
&= \sum_{\tau\in\mathbb{T}_t} [a(\sigma(\tau))X(\tau) + a(\sigma(\tau))\mu(\tau)X_1(\tau) + a(\sigma(\tau))X_2(\tau)\Delta W(\tau) - a(\tau)X(\tau)] \\
&= \sum_{\tau\in\mathbb{T}_t} (a(\sigma(\tau)) - a(\tau))X(\tau) + \sum_{\tau\in\mathbb{T}_t} \mu(\tau)a(\sigma(\tau))X_1(\tau) + \sum_{\tau\in\mathbb{T}_t} a(\sigma(\tau))X_2(\tau)\Delta W(\tau) \\
&= \int_{t_0}^{t} a^{\Delta}(\tau)X(\tau)\Delta\tau + \int_{t_0}^{t} a(\sigma(\tau))X_1(\tau)\Delta\tau + \int_{t_0}^{t} a(\sigma(\tau))X_2(\tau)\Delta W(\tau).
\end{aligned}$$

This shows (3.5). □

Next we give expectation $\mathbb{E}$ and variance $\mathbb{V}$ of stochastic dynamic integrals.

**Lemma 3.6.** *If $W$ is a Brownian motion and $X$ is an adapted stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then*

$$\mathbb{E}\left[\int_{t_0}^{t} X(\tau)\Delta W(\tau)\right] = 0 \tag{3.6}$$

*and*

$$\mathbb{V}\left[\int_{t_0}^{t} X(\tau)\Delta W(\tau)\right] = \int_{t_0}^{t} \mathbb{E}\left[X^2(\tau)\right]\Delta\tau \tag{3.7}$$

*for all $t \in \mathbb{T}$.*

*Proof.* To prove (3.6) we observe that (3.2) and (3.1) imply

$$\begin{aligned}
\mathbb{E}\left[\int_{t_0}^{t} X(\tau)\Delta W(\tau)\right] &= \mathbb{E}\left[\sum_{\tau\in\mathbb{T}_t} X(\tau)\Delta W(\tau)\right] \\
&= \sum_{\tau\in\mathbb{T}_t} \mathbb{E}[X(\tau)\Delta W(\tau)] \\
&= \sum_{\tau\in\mathbb{T}_t} \mathbb{E}[X(\tau)]\,\mathbb{E}[\Delta W(\tau)] = 0,
\end{aligned}$$

since $X(\tau)$ is $\mathcal{F}(\tau)$-measurable and $\Delta W(\tau)$ is independent of $\mathcal{F}(\tau)$. Next we observe that

$$\begin{aligned}\mathbb{E}\left[\left(\int_{t_0}^{t} X(\tau)\Delta W(\tau)\right)^2\right] &= \mathbb{E}\left[\left(\sum_{\tau\in\mathbb{T}_t} X(\tau)\Delta W(\tau)\right)^2\right] \\ &= \sum_{\tau_1\in\mathbb{T}_t}\sum_{\tau_2\in\mathbb{T}_t} \mathbb{E}\left[X(\tau_1)X(\tau_2)\Delta W(\tau_1)\Delta W(\tau_2)\right].\end{aligned}$$

Now if $\tau_1 < \tau_2$, then $\Delta W(\tau_2)$ is independent of $X(\tau_1)X(\tau_2)\Delta W(\tau_1)$. Thus,

$$\mathbb{E}\left[X(\tau_1)X(\tau_2)\Delta W(\tau_1)\Delta W(\tau_2)\right] = \mathbb{E}\left[X(\tau_1)X(\tau_2)\Delta W(\tau_1)\right]\mathbb{E}\left[\Delta W(\tau_2)\right] = 0.$$

Consequently, (2.2) and (3.1) imply

$$\begin{aligned}\mathbb{E}\left[\left(\int_{t_0}^{t} X(\tau)\Delta W(\tau)\right)^2\right] &= \sum_{\tau\in\mathbb{T}_t} \mathbb{E}\left[X^2(\tau)(\Delta W(\tau))^2\right] \\ &= \sum_{\tau\in\mathbb{T}_t} \mathbb{E}\left[X^2(\tau)\right]\mathbb{E}\left[(\Delta W(\tau))^2\right] \\ &= \sum_{\tau\in\mathbb{T}_t} \mathbb{E}\left[X^2(\tau)\right]\mu(\tau) = \int_{t_0}^{t} \mathbb{E}\left[X^2(\tau)\right]\Delta\tau,\end{aligned}$$

and this concludes the proof. □

# 4 Stochastic Dynamic Exponential

**Definition 4.1.** Let $W$ be Brownian motion on $\mathbb{T}$. Then we say an adapted stochastic process $A$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *stochastic regressive* (with respect to $W$) provided

$$1 + A(t)\Delta W(t) \neq 0 \quad \text{a.s.} \quad \text{for all} \quad t \in \mathbb{T}.$$

The set of stochastic regressive functions will be denoted by $\mathcal{R}_W$. We define the *stochastic circle plus* $\oplus_W$ on $\mathcal{R}_W$ by

$$A \oplus_W B := A + B + AB\Delta W \quad \text{on} \quad \mathbb{T}. \tag{4.1}$$

**Theorem 4.2.** *$(\mathcal{R}_W, \oplus_W)$ is an Abelian group.*

*Proof.* To prove closure under the addition $\oplus_W$, we note that, for $A, B \in \mathcal{R}_W$, $A \oplus_W B$ is a function from $\mathbb{T}$ to $\mathbb{R}$. It only remains to show that on $\mathbb{T}$, $A \oplus_W B \neq -1/\Delta W$ a.s., but this follows from

$$\begin{aligned}1 + (A \oplus_W B)\Delta W &= 1 + (A + B + AB\Delta W)\Delta W \\ &= (1 + A\Delta W)(1 + B\Delta W) \neq 0 \quad \text{a.s.}\end{aligned}$$

Hence, $\mathcal{R}_W$ is closed under the addition $\oplus_W$. Since

$$A \oplus_W 0 = 0 \oplus_W A = A,$$

0 is the additive identity for $\oplus_W$. For $A \in \mathcal{R}_W$, to find the additive inverse of $A$ under $\oplus_W$, we solve $A \oplus_W B = 0$ for $B$, i.e.,

$$B = -\frac{A}{1 + A\Delta W}.$$

That the associative law holds follows from

$$\begin{aligned}
(A \oplus_W B) \oplus_W C &= (A + B + AB\Delta W) \oplus_W C \\
&= (A + B + AB\Delta W) + C + (A + B + AB\Delta W)C\Delta W \\
&= A + (B + C + BC\Delta W) + A(B + C + BC\Delta W)\Delta W \\
&= A \oplus_W (B \oplus_W C)
\end{aligned}$$

for $A, B, C \in \mathcal{R}_W$. Hence, $(\mathcal{R}_W, \oplus_W)$ is a group. Since

$$A \oplus_W B = A + B + AB\Delta W = B + A + BA\Delta W = B \oplus_W A,$$

the commutative law holds, and hence $(\mathcal{R}_W, \oplus_W)$ is an Abelian group. $\square$

**Definition 4.3.** For $A \in \mathcal{R}_W$, the *stochastic circle minus* $\ominus_W$ on $\mathcal{R}_W$ is defined as

$$\ominus_W A := \frac{-A}{1 + A\Delta W} \quad \text{on} \quad \mathbb{T}, \tag{4.2}$$

and we also put

$$B \ominus_W A := B \oplus_W (\ominus_W A) \quad \text{on} \quad \mathbb{T}. \tag{4.3}$$

**Theorem 4.4.** *If $A, B \in \mathcal{R}_W$, then*

(1) $\ominus_W(\ominus_W A) = A$,

(2) $A \ominus_W B = \frac{A-B}{1+B\Delta W}$,

(3) $A \ominus_W A = 0$,

(4) $A \ominus_W B \in \mathcal{R}_W$,

(5) $\ominus_W(A \ominus_W B) = B \ominus_W A$,

(6) $\ominus_W(A \oplus_W B) = (\ominus_W A) \oplus_W (\ominus_W B)$.

*Proof.* Using (4.2), we observe that

$$\ominus_W(\ominus_W A) = \ominus_W\left(\frac{-A}{1 + A\Delta W}\right) = \frac{-\left(\frac{-A}{1+A\Delta W}\right)}{1 + \left(\frac{-A}{1+A\Delta W}\right)\Delta W} = A,$$

where on the first and second equality we have used (4.2). This shows (1). Next, from (4.2), (4.3) and (4.1), we have

$$A \ominus_W B = A \oplus_W \left(\frac{-B}{1 + B\Delta W}\right) = A + \left(\frac{-B}{1 + B\Delta W}\right) + A\left(\frac{-B}{1 + B\Delta W}\right)\Delta W = \frac{A - B}{1 + B\Delta W},$$

as claimed in (2), which also implies (3) immediately, and (2) also shows (4) as

$$1+(A\ominus_W B)\Delta W = 1+\frac{A-B}{1+B\Delta W}\Delta W = \frac{1+A\Delta W}{1+B\Delta W}\neq 0 \quad \text{a.s.}$$

since $A, B \in \mathcal{R}_W$. For (5), we observe that

$$\ominus_W(A\ominus_W B) = \ominus_W\left(\frac{A-B}{1+B\Delta W}\right) = \frac{-\left(\frac{A-B}{1+B\Delta W}\right)}{1+\left(\frac{A-B}{1+B\Delta W}\right)\Delta W} = \frac{B-A}{1+A\Delta W} = B\ominus_W A,$$

where on the first and last equality we have used part (2) and on the second equality we have used (4.2). Finally, to show (6), we observe that

$$\begin{aligned}
(\ominus_W A)\oplus_W(\ominus_W B) &= \left(\frac{-A}{1+A\Delta W}\right)\oplus_W\left(\frac{-B}{1+B\Delta W}\right)\\
&= \frac{-A}{1+A\Delta W}+\frac{-B}{1+B\Delta W}+\frac{AB\Delta W}{(1+A\Delta W)(1+B\Delta W)}\\
&= \frac{-(A+B+AB\Delta W)}{1+(A+B+AB\Delta W)\Delta W}\\
&= \frac{-(A\oplus_W B)}{1+(A\oplus_W B)\Delta W} = \ominus_W(A\oplus_W B),
\end{aligned}$$

where we have used (4.1), (4.2) and (2). □

**Definition 4.5.** If $t_0 \in \mathbb{T}$ and $A \in \mathcal{R}_W$, then the unique solution of

$$\Delta X = A(t)X\Delta W, \quad X(t_0) = 1, \quad t \in \mathbb{T} \tag{4.4}$$

is called the *stochastic exponential* and denoted by

$$X = E_A(\cdot, t_0). \tag{4.5}$$

Note that by Definition 3.3,

$$E_A(t,t_0) = 1+\int_{t_0}^{t} A(\tau)E_A(\tau,t_0)\Delta W(\tau) \quad \text{for all } t\in\mathbb{T}. \tag{4.6}$$

Using this and the following auxiliary result, we will present a closed-form expression for $E_A$ in Theorem 4.7 below.

**Lemma 4.6.** *Let $f, g : \mathbb{T} \to \mathbb{R}$. Then*

$$f(t) = f(t_0) + \sum_{\tau\in\mathbb{T}_t} f(\tau)g(\tau) \quad \textit{for all } t \geq t_0 \tag{4.7}$$

*if and only if*

$$f(t) = f(t_0)\prod_{\tau\in\mathbb{T}_t}[1+g(\tau)] \quad \textit{for all } t \geq t_0. \tag{4.8}$$

*Proof.* Assume (4.7). We prove (4.8) using the induction principle given in [1, Theorem 1.7]. We denote the statement in (4.8) by $S(t)$ and observe that $S(t_0)$ is trivially satisfied. Now, assuming that $S(t)$ holds, we have

$$\begin{aligned} f(\sigma(t)) &= f(t_0) + \sum_{\tau \in \mathbb{T}_{\sigma(t)}} f(\tau)g(\tau) \\ &= f(t_0) + f(t)g(t) + \sum_{\tau \in \mathbb{T}_t} f(\tau)g(\tau) \\ &= f(t)g(t) + f(t) = [1+g(t)]f(t) \\ &= [1+g(t)]f(t_0) \prod_{\tau \in \mathbb{T}_t} [1+g(\tau)] \\ &= f(t_0) \prod_{\tau \in \mathbb{T}_{\sigma(t)}} [1+g(\tau)], \end{aligned}$$

which proves $S(\sigma(t))$. Conversely, assume (4.8). Then

$$f(\sigma(t)) = f(t_0) \prod_{\tau \in \mathbb{T}_{\sigma(t)}} [1+g(\tau)] = f(t_0)[1+g(t)] \prod_{\tau \in \mathbb{T}_t} [1+g(\tau)] = f(t)[1+g(t)].$$

So $\Delta f(t) = f(t)g(t)$, and $f(t) - f(t_0) = \sum_{\tau \in \mathbb{T}_t} f(\tau)g(\tau)$, which proves (4.7). □

From Lemma 4.6, we have the following result.

**Theorem 4.7.** *$E_A(\cdot, t_0)$ defined in Definition 4.5 is given by*

$$E_A(t,t_0) = \prod_{\tau \in \mathbb{T}_t} [1 + A(\tau)\Delta W(\tau)]. \tag{4.9}$$

**Example 4.8.** We give the stochastic exponential for various time scales as follows.

1. If $\mathbb{T} = \mathbb{Z}$, then
$$E_A(t,t_0) = \prod_{\tau=t_0}^{t-1} [1 + A(\tau)(W(\tau+1) - W(\tau))].$$

2. If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then
$$E_A(t,t_0) = \prod_{\tau=\frac{t_0}{h}}^{\frac{t}{h}-1} [1 + A(h\tau)(W(h\tau+h) - W(h\tau))].$$

3. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then
$$E_A(t,t_0) = \prod_{\tau=\log_q t_0}^{\log_q t-1} \left[1 + A\left(q^\tau\right)\left(W\left(q^{\tau+1}\right) - W\left(q^\tau\right)\right)\right].$$

*Remark* 4.9. If $\mathbb{T} = \mathbb{R}$ and $A(t) = a(t)$ is deterministic, then we define $E_a(\cdot, t_0)$ as the solution of the stochastic differential problem

$$\mathrm{d}X = a(t)X\mathrm{d}W, \quad X(t_0) = 1, \quad t \in \mathbb{R},$$

which is well known [4, 5] to be

$$E_a(t,t_0) = \exp\left(-\frac{1}{2}\int_{t_0}^{t} a^2(s)\mathrm{d}s + \int_{t_0}^{t} a(s)\mathrm{d}W(s)\right), \quad t \in \mathbb{R}. \tag{4.10}$$

Although our theory and formulas presented below are proved only for isolated time scales, we may see that all of our formulas, using (4.10), remain true for $\mathbb{T} = \mathbb{R}$.

**Theorem 4.10.** *If $A, B \in \mathcal{R}_W$, then*

(1) $E_A(\sigma(t),t_0) = (1 + A(t)\Delta W(t))E_A(t,t_0)$,

(2) $\frac{1}{E_A(t,t_0)} = E_{\ominus_W A}(t,t_0)$,

(3) $E_A(t,t_0)E_B(t,t_0) = E_{A\oplus_W B}(t,t_0)$,

(4) $\frac{E_A(t,t_0)}{E_B(t,t_0)} = E_{A\ominus_W B}(t,t_0)$.

*Proof.* We use Theorem 4.7. First,

$$\begin{aligned} E_A(\sigma(t),t_0) &= \prod_{\tau\in\mathbb{T}_{\sigma(t)}} [1 + A(\tau)\Delta W(\tau)] \\ &= (1 + A(t)\Delta W(t)) \prod_{\tau\in\mathbb{T}_t} [1 + A(\tau)\Delta W(\tau)] \\ &= (1 + A(t)\Delta W(t))\, E_A(t,t_0) \end{aligned}$$

shows (1), and (2) follows from

$$\begin{aligned} E_{\ominus_W A}(t,t_0) &= \prod_{\tau\in\mathbb{T}_t} [1 + (\ominus_W A)(\tau)\Delta W(\tau)] \\ &= \prod_{\tau\in\mathbb{T}_t} \left[1 - \frac{A(\tau)}{1 + A(\tau)\Delta W(\tau)}\Delta W(\tau)\right] \\ &= \frac{1}{\prod_{\tau\in\mathbb{T}_t} [1 + A(\tau)\Delta W(\tau)]} \\ &= \frac{1}{E_A(t,t_0)}. \end{aligned}$$

For (3), we observe that

$$\begin{aligned} E_A(t,t_0)E_B(t,t_0) &= \prod_{\tau\in\mathbb{T}_t} [1 + A(\tau)\Delta W(\tau)] \prod_{\tau\in\mathbb{T}_t} [1 + B(\tau)\Delta W(\tau)] \\ &= \prod_{\tau\in\mathbb{T}_t} \left[1 + A(\tau)\Delta W(\tau) + B(\tau)\Delta W(\tau) + A(\tau)B(\tau)(\Delta W(\tau))^2\right] \\ &= \prod_{\tau\in\mathbb{T}_t} [1 + (A(\tau) + B(\tau) + A(\tau)B(\tau)\Delta W(\tau))\Delta W(\tau)] \\ &= \prod_{\tau\in\mathbb{T}_t} [1 + (A \oplus_W B)(\tau)\Delta W(\tau)] \\ &= E_{A\oplus_W B}(t,t_0), \end{aligned}$$

and (4) follows from

$$\begin{aligned} E_{A \ominus_W B}(t,t_0) &= \prod_{\tau \in \mathbb{T}_t} \left[1 + \left(\frac{A(\tau)-B(\tau)}{1+B(\tau)\Delta W(\tau)}\right)\Delta W(\tau)\right] \\ &= \frac{\prod_{\tau \in \mathbb{T}_t} [1+A(\tau)\Delta W(\tau)]}{\prod_{\tau \in \mathbb{T}_t} [1+B(\tau)\Delta W(\tau)]} \\ &= \frac{E_A(t,t_0)}{E_B(t,t_0)}. \end{aligned}$$

This completes the proof. □

We conclude this section by offering formulas for the expectation and variance of the stochastic exponential in Theorem 4.12 below. We first present the following auxiliary result.

**Lemma 4.11.** *If $A$ is adapted, then $E_A(\cdot,t_0)$ is adapted.*

*Proof.* We use an induction argument given in [1, Theorem 1.7]. Denote the statement that $E_A(t,t_0)$ is $\mathcal{F}(t)$-measurable by $S(t)$. It is clear that $S(t_0)$ is true. Assume that $S(t)$ holds. By Theorem 4.10 (1) we have

$$E_A(\sigma(t),t_0) = [1+A(t)\Delta W(t)]E_A(t,t_0).$$

Now $A(t)$ is $\mathcal{F}(t)$-measurable and $\Delta W(t)$ is $\mathcal{F}(\sigma(t))$-measurable imply that $1+A(t)\Delta W(t)$ is $\mathcal{F}(\sigma(t))$-measurable. Since $E_A(t,t_0)$ is $\mathcal{F}(t)$-measurable by assumption, we have that $E_A(\sigma(t),t_0)$ is $\mathcal{F}(\sigma(t))$-measurable, which proves $S(\sigma(t))$. □

**Theorem 4.12.** *If $E_A(\cdot,t_0)$ is given as in Definition 4.5 and $A$ is adapted, then*

$$\mathbb{E}[E_A(t,t_0)] = 1 \tag{4.11}$$

*and if, moreover, $A$ and $E_A(t,t_0)$ are independent, then*

$$\mathbb{V}[E_A(t,t_0)] = e_{\mathbb{E}[A^2]}(t,t_0) - 1. \tag{4.12}$$

*Proof.* Taking expectation on both sides of (4.6), i.e., of

$$E_A(t,t_0) = 1 + \int_{t_0}^{t} A(\tau)E_A(\tau,t_0)\Delta W(\tau),$$

we have

$$\mathbb{E}[E_A(t,t_0)] = 1 + \mathbb{E}\left[\int_{t_0}^{t} A(\tau)E_A(\tau,t_0)\Delta W(\tau)\right] = 1, \tag{4.13}$$

where on the second equality we have used (3.6) and Lemma 4.11. Likewise,

$$\begin{aligned} \mathbb{E}\left[E_A^2(t,t_0)\right] &= 1 + 2\mathbb{E}\left[\int_{t_0}^{t} A(\tau)E_A(\tau,t_0)\Delta W(\tau)\right] + \mathbb{V}\left[\int_{t_0}^{t} A(\tau)E_A(\tau,t_0)\Delta W(\tau)\right] \\ &= 1 + \int_{t_0}^{t} \mathbb{E}\left[(A(\tau)E_A(\tau,t_0))^2\right]\Delta\tau \\ &= 1 + \int_{t_0}^{t} \mathbb{E}\left[A^2(\tau)\right]\mathbb{E}\left[E_A^2(\tau,t_0)\right]\Delta\tau, \end{aligned}$$

where we have used (3.6), (3.7) and Lemma 4.11. If we take $y(t) = \mathbb{E}\left[E_A^2(t,t_0)\right]$, then $y$ satisfies the initial value problem $y^\Delta = \mathbb{E}\left[A^2(t)\right] y$, $y(t_0) = 1$, whose solution from Theorem 2.1 is $y(t) = e_{\mathbb{E}[A^2]}(t,t_0)$. Using this fact, we conclude that

$$\mathbb{E}\left[E_A^2(t,t_0)\right] = e_{\mathbb{E}[A^2]}(t,t_0),$$

and we have

$$\mathbb{V}\left[E_A(t,t_0)\right] = \mathbb{E}\left[E_A^2(t,t_0)\right] - \left(\mathbb{E}\left[E_A(t,t_0)\right]\right)^2 = e_{\mathbb{E}[A^2]}(t,t_0) - 1,$$

as claimed. □

## 5 Dynamic Geometric Brownian Motion

In this section, we construct and study the properties of geometric Brownian motion on a time scale $\mathbb{T}$. Let us consider the homogeneous linear stochastic dynamic equation

$$\Delta X = a(t)X\Delta t + b(t)X\Delta W, \tag{5.1}$$

where $a, b : \mathbb{T} \to \mathbb{R}$, $X$ is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $W$ is a Brownian motion indexed by $\mathbb{T}$. Obviously, $X(t) \equiv 0$ is a solution of (5.1).

**Theorem 5.1.** *If $t_0 \in \mathbb{T}$, $a \in \mathcal{R}$ and $\frac{b}{1+\mu a} \in \mathcal{R}_W$, then the unique solution of*

$$\Delta X = a(t)X\Delta t + b(t)X\Delta W, \quad X(t_0) = X_0 \tag{5.2}$$

*is given by*

$$X = X_0 e_a(\cdot,t_0) E_{\frac{b}{1+\mu a}}(\cdot,t_0). \tag{5.3}$$

*Proof.* Let $X(t) = X_0 e_a(t,t_0) E_{\frac{b}{1+\mu a}}(t,t_0)$. Then $X(t_0) = X_0$ and

$$\begin{aligned}
&X(t_0) + \int_{t_0}^t a(\tau)X(\tau)\Delta\tau + \int_{t_0}^t b(\tau)X(\tau)\Delta W(\tau) \\
&= X_0 + X_0 \int_{t_0}^t a(\tau)e_a(\tau,t_0)E_{\frac{b}{1+\mu a}}(\tau,t_0)\Delta\tau + X_0 \int_{t_0}^t b(\tau)e_a(\tau,t_0)E_{\frac{b}{1+\mu a}}(\tau,t_0)\Delta W(\tau) \\
&= X_0\left\{1 + \int_{t_0}^t e_a^\Delta(\tau,t_0)E_{\frac{b}{1+\mu a}}(\tau,t_0)\Delta\tau + \int_{t_0}^t e_a(\sigma(\tau),t_0)\frac{b(\tau)}{1+\mu(\tau)a(\tau)}E_{\frac{b}{1+\mu a}}(\tau,t_0)\Delta W(\tau)\right\} \\
&= X_0\left\{1 + e_a(t,t_0)E_{\frac{b}{1+\mu a}}(t,t_0) - e_a(t_0,t_0)E_{\frac{b}{1+\mu a}}(t_0,t_0)\right\} \\
&= X_0 e_a(t,t_0)E_{\frac{b}{1+\mu a}}(t,t_0) = X(t),
\end{aligned}$$

where on the third equality we have used Lemma 3.5 with $X_1 \equiv 0$. For the converse assume that $X$ solves (5.2). Assuming $X_0 \neq 0$, define $Y = \frac{X}{X_0} e_{\ominus a}(\cdot, t_0)$. Then $Y(t_0) = 1$ and

$$
\begin{aligned}
1+\int_{t_0}^{t} & \frac{b(\tau)}{1+\mu(\tau)a(\tau)} Y(\tau)\Delta W(\tau) = 1+\frac{1}{X_0}\int_{t_0}^{t} \frac{b(\tau)}{1+\mu(\tau)a(\tau)} e_{\ominus a}(\tau,t_0)X(\tau)\Delta W(\tau) \\
= \quad & 1+\frac{1}{X_0}\int_{t_0}^{t} e_{\ominus a}(\sigma(\tau),t_0)b(\tau)X(\tau)\Delta W(\tau) \\
= \quad & 1+\frac{1}{X_0}\left\{ e_{\ominus a}(t,t_0)X(t) - e_{\ominus a}(t_0,t_0)X(t_0) - \int_{t_0}^{t} e^{\Delta}_{\ominus a}(\tau,t_0)X(\tau)\Delta\tau \right. \\
& \qquad \left. - \int_{t_0}^{t} e_{\ominus a}(\sigma(\tau),t_0)a(\tau)X(\tau)\Delta\tau \right\} \\
= \quad & 1+\frac{1}{X_0}\left\{ e_{\ominus a}(t,t_0)X(t) - X_0 - \int_{t_0}^{t} (\ominus a)(\tau)e_{\ominus a}(\tau,t_0)X(\tau)\Delta\tau \right. \\
& \qquad \left. - \int_{t_0}^{t} (1+\mu(\tau)(\ominus a)(\tau))e_{\ominus a}(\tau,t_0)a(\tau)X(\tau)\Delta\tau \right\} \\
= \quad & 1+\frac{1}{X_0}\left\{ e_{\ominus a}(t,t_0)X(t) - X_0 - \int_{t_0}^{t} \left( (\ominus a)(\tau) + \frac{a(\tau)}{1+\mu(\tau)a(\tau)} \right) e_{\ominus a}(\tau,t_0)X(\tau)\Delta\tau \right\} \\
= \quad & 1+\frac{X(t)}{X_0} e_{\ominus a}(t,t_0) - 1 \\
= \quad & \frac{X(t)}{X_0} e_{\ominus a}(t,t_0) = Y(t),
\end{aligned}
$$

where on the third equality we have used Lemma 3.5. Hence $Y = E_{\frac{b}{1+\mu a}}(\cdot, t_0)$ and thus $X = X_0 e_a(\cdot, t_0) E_{\frac{b}{1+\mu a}}(\cdot, t_0)$. □

In the proof above we have not used Itô's lemma which is usually employed when $\mathbb{T} = \mathbb{R}$.

**Example 5.2.** Using Example 4.8 and Theorem 5.1, we now present the solution of (5.2) on various time scales.

1. If $\mathbb{T} = \mathbb{Z}$, then

$$X(t) = X_0 \prod_{\tau=t_0}^{t-1} \left[1 + a(\tau) + b(\tau)(W(\tau+1) - W(\tau))\right],$$

   if $a(\tau) \neq -1$ for all $\tau \in \mathbb{Z}$.

2. If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then

$$X(t) = X_0 \prod_{\tau=\frac{t_0}{h}}^{\frac{t}{h}-1} \left[1 + ha(h\tau) + b(h\tau)(W(h\tau+h) - W(h\tau))\right],$$

   if $a(h\tau) \neq -\frac{1}{h}$ for all $\tau \in \mathbb{Z}$.

3. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then

$$X(t) = X_0 \prod_{\tau=\log_q t_0}^{\log_q t - 1} \left[1 + (q-1)q^\tau a(q^\tau) + b(q^\tau)\left(W(q^{\tau+1}) - W(q^\tau)\right)\right],$$

if $q^\tau a(q^\tau) \neq \frac{1}{1-q}$ for all $\tau \in \mathbb{N}_0$.

*Remark* 5.3. If $\mathbb{T} = \mathbb{R}$, then (5.2) becomes

$$\mathrm{d}X = a(t)X\mathrm{d}t + b(t)X\mathrm{d}W, \quad X(t_0) = 1 \tag{5.4}$$

whose solution is well known [4, 5] to be

$$X(t) = \exp\left(\int_{t_0}^t \left(a(s) - \frac{1}{2}b^2(s)\right)\mathrm{d}s + \int_{t_0}^t b(s)\mathrm{d}W(s)\right). \tag{5.5}$$

Note that in this case (5.3) becomes

$$\begin{aligned} X(t) &= e_a(t,t_0)E_b(t,t_0) \\ &= \exp\left(\int_{t_0}^t a(s)\mathrm{d}s\right)\exp\left(-\frac{1}{2}\int_{t_0}^t b^2(s)\mathrm{d}s + \int_{t_0}^t b(s)\mathrm{d}W(s)\right) \\ &= \exp\left(\int_{t_0}^t \left(a(s) - \frac{1}{2}b^2(s)\right)\mathrm{d}s + \int_{t_0}^t b(s)\mathrm{d}W(s)\right), \end{aligned}$$

which is the same as (5.5). Hence our solution formula (5.3) is valid for $\mathbb{T} = \mathbb{R}$ as well.

## 6 A Stock Price Model

Let $S(t)$ denote the price of a stock at time $t$ and $S(t_0) = S_0$ the current price of the stock. Then the evolution of $S(t)$ in time may be modeled by supposing that $\Delta S/S$, the relative change in price, evolves according to the stochastic dynamic equation

$$\frac{\Delta S}{S} = \alpha(t)\Delta t + \beta(t)\Delta W, \quad S(t_0) = S_0 > 0$$

such that $\alpha \in \mathcal{R}$ and $\frac{\beta}{1+\mu\alpha} \in \mathcal{R}_W$, called the *drift* and the *volatility* of the stock. Then

$$\Delta S = \alpha(t)S\Delta t + \beta(t)S\Delta W, \tag{6.1}$$

which is same as (5.1), and so by (5.3) we have

$$S(t) = S_0 e_\alpha(t,t_0)E_{\frac{\beta}{1+\mu\alpha}}(t,t_0). \tag{6.2}$$

Thus,

$$\mathbb{E}[S(t)] = \mathbb{E}\left[S_0 e_\alpha(t,t_0)E_{\frac{\beta}{1+\mu\alpha}}(t,t_0)\right] = S_0 e_\alpha(t,t_0)\mathbb{E}\left[E_{\frac{\beta}{1+\mu\alpha}}(t,t_0)\right] = S_0 e_\alpha(t,t_0), \tag{6.3}$$

where on the third equality we have used (4.11) of Theorem 4.12. We can also arrive at (6.3) by rewriting (6.1), using Definition 3.3, as

$$S(t) = S(t_0) + \int_{t_0}^{t} \alpha(\tau)S(\tau)\Delta\tau + \int_{t_0}^{t} \beta(\tau)S(\tau)\Delta W(\tau)$$

and calculating

$$\begin{aligned}\mathbb{E}[S(t)] &= \mathbb{E}[S(t_0)] + \mathbb{E}\left[\int_{t_0}^{t} \alpha(\tau)S(\tau)\Delta\tau\right] + \mathbb{E}\left[\int_{t_0}^{t} \beta(\tau)S(\tau)\Delta W(\tau)\right] \\ &= S_0 + \int_{t_0}^{t} \alpha(\tau)\mathbb{E}[S(\tau)]\Delta\tau,\end{aligned}$$

where we have (note that $S$ is adapted because of (6.2) and Lemma 4.11) used (3.6). If we take $y(t) = \mathbb{E}[S(t)]$, then this is a first-order homogeneous linear dynamic equation of the form $y^\Delta = \alpha(t)y$, $y(t_0) = y_0$, whose solution by Theorem 2.1 is $y(t) = e_\alpha(t,t_0)y_0$. Using this fact, we conclude that

$$\mathbb{E}[S(t)] = S_0 e_\alpha(t,t_0). \tag{6.4}$$

For the variance of the stock price, we observe that

$$\begin{aligned}\mathbb{V}[S(t)] &= \mathbb{E}[S^2(t)] - (\mathbb{E}[S(t)])^2 \\ &= S_0^2 e_\alpha^2(t,t_0)\mathbb{E}\left[E^2_{\frac{\beta}{1+\mu\alpha}}(t,t_0)\right] - \left(S_0 e_\alpha(t,t_0)\mathbb{E}\left[E_{\frac{\beta}{1+\mu\alpha}}(t,t_0)\right]\right)^2 \\ &= S_0^2 e_\alpha^2(t,t_0)\mathbb{V}\left[E_{\frac{\beta}{1+\mu\alpha}}(t,t_0)\right] \\ &= S_0^2 e_\alpha^2(t,t_0)\left(e_{\frac{\beta^2}{(1+\mu\alpha)^2}}(t,t_0) - 1\right),\end{aligned} \tag{6.5}$$

where on the third equality we have used (4.12) of Theorem 4.12.

**Example 6.1.** From (6.4), the expected values of the stock price at time $t$ for various time scales are given as follows.

1. If $\mathbb{T} = \mathbb{Z}$, then

   $$\mathbb{E}[S(t)] = S_0 \prod_{\tau=t_0}^{t-1}(1+\alpha(\tau)) \quad \text{if } \alpha(\tau) \neq -1 \text{ for all } \tau \in \mathbb{Z}$$

   and

   $$\mathbb{E}[S(t)] = S_0(1+\alpha)^{t-t_0} \quad \text{for constant } \alpha \neq -1.$$

2. If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then

   $$\mathbb{E}[S(t)] = S_0 \prod_{\tau=\frac{t_0}{h}}^{\frac{t}{h}-1}(1+h\alpha(h\tau)) \quad \text{if } \alpha(h\tau) \neq -\tfrac{1}{h} \text{ for all } \tau \in \mathbb{Z}$$

   and

   $$\mathbb{E}[S(t)] = S_0(1+h\alpha)^{\frac{t-t_0}{h}} \quad \text{for constant } \alpha \neq -\tfrac{1}{h}.$$

3. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then

$$\mathbb{E}[S(t)] = S_0 \prod_{\tau=\log_q t_0}^{\log_q t-1} (1+(q-1)q^{\tau}\alpha(q^{\tau})) \quad \text{if } q^{\tau}\alpha(q^{\tau}) \neq \tfrac{1}{1-q} \text{ for all } \tau \in \mathbb{N}_0.$$

*Remark* 6.2. If $\mathbb{T} = \mathbb{R}$, then it is well known [4, 5] that

$$\mathbb{E}[S(t)] = S_0 e^{\int_{t_0}^{t} \alpha(\tau)\mathrm{d}\tau} \quad \text{for continuous } \alpha$$

and

$$\mathbb{E}[S(t)] = S_0 e^{\alpha(t-t_0)} \quad \text{for constant } \alpha.$$

Note that these formulas are matching with our formula (6.4), and hence (6.4) is valid for $\mathbb{T} = \mathbb{R}$ also.

**Example 6.3.** From (6.5), the variances of the stock price at time $t$ for various time scales are given as follows.

1. If $\mathbb{T} = \mathbb{Z}$, then

$$\mathbb{V}[S(t)] = S_0^2 \left[\prod_{\tau=t_0}^{t-1} (1+\alpha(\tau))^2\right] \left[\prod_{\tau=t_0}^{t-1} \left(1+\frac{\beta^2(\tau)}{(1+\alpha(\tau))^2}\right) - 1\right]$$

if $\alpha(\tau) \neq -1$ for all $\tau \in \mathbb{Z}$ and

$$\mathbb{V}[S(t)] = S_0^2 (1+\alpha)^{2(t-t_0)} \left[\left(1+\frac{\beta^2}{(1+\alpha)^2}\right)^{t-t_0} - 1\right]$$

for constants $\beta$ and $\alpha \neq -1$.

2. If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then

$$\mathbb{V}[S(t)] = S_0^2 \left[\prod_{\tau=\frac{t_0}{h}}^{\frac{t}{h}-1} (1+h\alpha(h\tau))^2\right] \left[\prod_{\tau=\frac{t_0}{h}}^{\frac{t}{h}-1} \left(1+\frac{h\beta^2(h\tau)}{(1+h\alpha(h\tau))^2}\right) - 1\right]$$

if $\alpha(h\tau) \neq -\frac{1}{h}$ for all $\tau \in \mathbb{Z}$ and

$$\mathbb{V}[S(t)] = S_0^2 (1+h\alpha)^{\frac{2(t-t_0)}{h}} \left[\left(1+\frac{h\beta^2}{(1+h\alpha)^2}\right)^{\frac{t-t_0}{h}} - 1\right]$$

for constants $\beta$ and $\alpha \neq -\frac{1}{h}$.

3. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then

$$\begin{aligned} \mathbb{V}[S(t)] &= S_0^2 \prod_{\tau=\log_q t_0}^{\log_q t-1} \Big((1+(q-1)q^{\tau}\alpha(q^{\tau}))^2 + (q-1)q^{\tau}\beta^2(q^{\tau})\Big) \\ &\quad - S_0^2 \prod_{\tau=\log_q t_0}^{\log_q t-1} (1+(q-1)q^{\tau}\alpha(q^{\tau}))^2 \end{aligned}$$

if $q^{\tau}\alpha(q^{\tau}) \neq \frac{1}{1-q}$ for all $\tau \in \mathbb{N}_0$.

*Remark* 6.4. If $\mathbb{T} = \mathbb{R}$, then it is well known [4, 5] that

$$\mathbb{V}[S(t)] = S_0^2 e^{2\int_{t_0}^t \alpha(\tau)d\tau} \left[ e^{\int_{t_0}^t \beta^2(\tau)d\tau} - 1 \right]$$

for continuous $\alpha$ and $\beta$, and

$$\mathbb{V}[S(t)] = S_0^2 e^{2\alpha(t-t_0)} \left[ e^{\beta^2(t-t_0)} - 1 \right]$$

for constant $\alpha$ and $\beta$. Note that these formulas are matching with our formula (6.5), and hence (6.5) is valid for $\mathbb{T} = \mathbb{R}$ also.

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