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An application of time scales to economics

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Abstract

Economics is a discipline in which there appears to be many opportunities for applications of time scales. The time scales approach will not only unify the standard discrete and continuous models in economics, but also, for example, allows for payments which arrive at unequally spaced points in time. We present a dynamic optimization problem from economics, construct a time scales model, and apply calculus of variations to derive a solution. Time scale calculus would allow exploration of a variety of situations in economics.

Keywords: Calculus of variations; Time scales; Nabla derivative; Dynamic models; Optimization

1. Introduction

Time scale calculus is a relatively new theory (initiated in [1]) that unites the two approaches of dynamic modelling: difference and differential equations. In principle, these two approaches are special cases of a more general theory of time scale calculus. Time scale calculus theory is applicable to any field in which dynamic processes can be described with discrete or continuous models. Because many economic models are dynamic models, the results of time scale calculus are directly applicable to economics as well.

Economics is an ideal discipline for applications of time scales. Standard dynamic economic models are set up in either continuous or discrete time. For example, in a discrete model, a consumer receives some income in a time period and decides how much to consume and save during that same period. So, all decisions are assumed to be made at evenly spaced intervals.

The time scales approach to the above maximization problem is much more flexible and realistic. For example, a consumer receives income at one point in time, asset holdings are adjusted at a different point in time, and consumption takes place at yet another point in time. Moreover, consumption and saving decisions can be modeled to occur with arbitrary, time-varying frequency. It is hard to overestimate the advantages of such an approach over the discrete or continuous models used in economics. Time scale calculus would allow exploration of a variety of situations in which timing of the decisions impacts the decisions themselves.

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In this paper, the reader will find an economic application of the ∇-derivative and ∇-integral, first initiated by Atici and Guseinov [2]. The paper is organized as follows. In Section 2, we present some preliminary results for the ∇-derivative and nabla exponential function. In Section 3, we state and prove some basic theorems on calculus of variations for the time scale case. We want to point out that we used the nabla notion (i.e. ∇-derivative, ∇-integral, nabla exponential function) in this section, since our economics model in Section 4 relies heavily on this. The calculus of variation has been studied with the delta notion in the papers by Bohner [3] and Hilscher and Zeidan [4]. To demonstrate how the theory developed in Section 3 can be used in economics, Section 4 sets up a simple representative agent without uncertainty to show how this type of model is solved in the time scale setting.

We refer to the books for further reading on calculus of variations for the continuous case [5,6] and the discrete case [7]. For dynamic models in economics, we refer to the books [8–11].

2. Basic definitions on time scales

For our purposes, we let $\mathbb{T}$ be a time scale (a closed subset of $\mathbb{R}$), $[a, b]$ be the closed and bounded interval in $\mathbb{T}$, i.e., $[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}$ and $a, b \in \mathbb{T}$. For the reader's convenience, we state a few basic definitions on a time scale $\mathbb{T}$ [12].

Obviously, a time scale $\mathbb{T}$ may or may not be connected. Therefore we have the concept of *forward* and *backward jump operators* as follows. Define $\sigma, \rho : \mathbb{T} \mapsto \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, then $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense*, *left-scattered*, respectively. The set $\mathbb{T}_\kappa$ which is derived from $\mathbb{T}$ is as follows: if $\mathbb{T}$ has a right-scattered minimum t_1, then $\mathbb{T}_\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. We also define the *backwards graininess function* $\nu : \mathbb{T}_\kappa \mapsto [0, \infty)$ as $\nu(t) = t - \rho(t)$. If $f : \mathbb{T} \mapsto \mathbb{R}$ is a function, we define the function $f^\rho : \mathbb{T}_\kappa \mapsto \mathbb{R}$ by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}_\kappa$ and $\sigma^0(t) = \rho^0(t) = t$.

Definition 2.1. If $f : \mathbb{T} \mapsto \mathbb{R}$ is a function and $t \in \mathbb{T}_\kappa$, then we define the *nabla derivative* of f at a point t to be the number $f^\nabla(t)$ (provided it exists) with the property that, for each $\varepsilon > 0$, there is a neighborhood of U of t such that

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|,$$

for all $s \in U$.

Note that in the case $\mathbb{T} = \mathbb{R}$, then $f^\nabla(t) = f'(t)$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\nabla(t) = \nabla f(t) = f(t) - f(t-1)$.

Definition 2.2. A function $F : \mathbb{T} \longrightarrow \mathbb{R}$ we call a *nabla-antiderivative* of $f : \mathbb{T} \longrightarrow \mathbb{R}$ provided that $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}_k$. We then define the Cauchy ∇-integral from a to t of f by

$$\int_a^t f(s)\nabla s = F(t) - F(a) \quad \text{for all } t \in \mathbb{T}.$$

Note that in the case $\mathbb{T} = \mathbb{R}$ we have

$$\int_a^b f(t)\nabla t = \int_a^b f(t)\mathrm{d}t,$$

and in the case $\mathbb{T} = \mathbb{Z}$ we have

$$\int_a^b f(t)\nabla t = \sum_{k=a+1}^{b} f(k),$$

where $a, b \in \mathbb{T}$ with $a \leq b$.

Definition 2.3. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is *left-dense continuous* (or *ld-continuous*) provided that it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist at right-dense points of $\mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous if and only if f is continuous.

Definition 2.4. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is ν-regressive if

$$1 - \nu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}_\kappa.$$

Define the ν-regressive class of functions on $\mathbb{T}_\kappa$ to be

$$\mathcal{R}_\nu = \{p : \mathbb{T} \longrightarrow \mathbb{R} : p \text{ is ld-continuous and } \nu\text{-regressive}\}.$$

Definition 2.5. If $p \in \mathcal{R}_\nu$, then the *nabla exponential* function is defined by

$$\hat{e}_p(t,s) := \exp\left(\int_s^t \hat{\xi}_{\nu(\tau)}(p(\tau))\nabla\tau\right) \quad \text{for } s, t \in \mathbb{T},$$

where the ν-cylinder transformation $\hat{\xi}_\nu$ is as in [13, p. 49].

Note that in the case $\mathbb{T} = \mathbb{R}$, then $\hat{e}_\alpha(t,s) = e^{\alpha(t-s)}$, and if $\mathbb{T} = \mathbb{Z}$, then $\hat{e}_\alpha(t,s) = \left(\frac{1}{1-\alpha}\right)^{t-s}$, where $\alpha \in \mathbb{R} \setminus \{1\}$.

Many nice properties and examples of the nabla exponential function can be found in the book by Martin Bohner and Allan Peterson [13, Chapter 3].

For proof of the next theorem, we refer to Theorem 1.90 with Δ-derivative in [12].

Theorem 2.6 (*Chain Rule*). *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable and the formula*

$$(f \circ g)^\nabla(t) = \left\{\int_0^1 f'(g(t) + h\nu(t)g^\nabla(t))\mathrm{d}h\right\} g^\nabla(t)$$

holds.

The following lemma is crucial in the proof of Theorem 3.1 in the next section.

Lemma 2.1. *If $f(t)$ is continuous on $[\rho(a), b]$, where $\rho(a) < b$, and if*

$$\int_{\rho(a)}^b f(t)g(t)\nabla t = 0$$

for every function $g(t) \in C[\rho(a), b]$ with $g(\rho(a)) = g(b) = 0$, then $f(t) = 0$ for $t \in [\rho(a), b]$.

Proof. Suppose that the function $f(t)$ is nonzero, say positive, at some point in $[\rho(a), b]$. Without loss of generality, let's choose $[t_1, t_2] \subsetneq [\rho(a), b]$. Assume that $f(t_0) > 0$ at $t_0 \in [t_1, t_2]$. If we set

$$g(t) := (t - t_1 + \nu(t_0))(t_2 - t + \mu(t_0))$$

for $t \in [t_1, t_2]$ and $g(t) = 0$ otherwise.

Thus, we obtain a contradiction

$$\int_{\rho(a)}^b f(t)g(t)\nabla t = \int_{t_1}^{t_2} f(t)(t - t_1 + \nu(t_0))(t_2 - t + \mu(t_0))\nabla t > 0.$$

This completes the proof of the lemma. $\square$

3. Main results

Assume that $L(t, u, v)$ is a class C^2 function of (u, v) for each $t \in [\rho^2(a), \rho(b)] \subseteq \mathbb{T}$. Let $y \in C^1[\rho^2(a), \rho(b)]$ with $y(\rho^2(a)) = A$, $y(\rho(b)) = B$, where

$$C^1[\rho^2(a), \rho(b)] = \{y : [\rho^2(a), \rho(b)] \to \mathbb{R} \mid y^\nabla \text{ is continuous on } [\rho^2(a), \rho(b)]_\kappa\}.$$

The simplest variational problem is to extremize (maximize or minimize)

$$J[y] := \int_{\rho^2(a)}^{\rho(b)} L(t, y(\rho(t)), y^{\nabla}(t))\nabla t.$$

We say that $y_0 \in C^1[\rho^2(a), \rho(b)]$ minimizes this variation problem if

$$J[y_0] \leq J[y]$$

for all $y \in C^1[\rho^2(a), \rho(b)]$. We say J has a local minimum at y_0 provided that there is a $\delta > 0$ such that

$$J[y_0] \leq J[y]$$

for all $y \in C^1[\rho^2(a), \rho(b)]$ with $\|y - y_0\| < \delta$. Here we consider the norm

$$\|y\| = \max_{t\in[\rho^2(a),\rho(b)]} |y(t)| + \max_{t\in[\rho^2(a),\rho(b)]_\kappa} |y^{\nabla}(t)|.$$

In this section we develop necessary conditions for the simplest variational problem. Now let $h : [\rho^2(a), \rho(b)] \to \mathbb{R}$ be any admissible variation, i.e., $h \in C^1[\rho^2(a), \rho(b)]$ with $h(\rho^2(a)) = h(\rho(b)) = 0$. Assume that this variational problem has a local extremum at y.

Then we define

$$\varphi(\epsilon) := J[y(t) + \epsilon h(t)],$$

where $-\infty < \epsilon < \infty$.

Since φ has a local extremum at $\epsilon = 0$, we have that

$$\varphi'(0) = 0$$
$$\varphi''(0) \geq 0 \ (\leq 0)$$

in the local minimum (maximum) case.

Next we consider

$$\varphi(\epsilon) = \int_{\rho^2(a)}^{\rho(b)} L(t, y(\rho(t)) + \epsilon h(\rho(t)), y^{\nabla}(t) + \epsilon h^{\nabla}(t))\nabla t.$$

Differentiating with respect to ϵ, we have

$$\begin{aligned}\varphi'(\epsilon) &= \int_{\rho^2(a)}^{\rho(b)} \frac{\mathrm{d}}{\mathrm{d}\epsilon} L(t, y(\rho(t)) + \epsilon h(\rho(t)), y^{\nabla}(t) + \epsilon h^{\nabla}(t))\nabla t \\ &= \int_{\rho^2(a)}^{\rho(b)} \{L_u(t, y(\rho(t)) + \epsilon h(\rho(t)), y^{\nabla}(t) + \epsilon h^{\nabla}(t))h(\rho(t)) \\ &\quad + L_v(t, y(\rho(t)) + \epsilon h(\rho(t)), y^{\nabla}(t) + \epsilon h^{\nabla}(t))h^{\nabla}(t)\}\nabla t.\end{aligned}$$

Hence, we obtain

$$\varphi'(0) = \int_{\rho^2(a)}^{\rho(b)} \{L_{y^\rho}(t, y^\rho(t), y^{\nabla}(t))h^\rho(t) + L_{y^\nabla}(t, y^\rho(t), y^{\nabla}(t))h^{\nabla}(t)\}\nabla t.$$

The integral

$$\int_{\rho^2(a)}^{\rho(b)} \{L_{y^\rho}(t, y^\rho, y^{\nabla})h^\rho + L_{y^\nabla}(t, y^\rho, y^{\nabla})h^{\nabla}\}\nabla t$$

gives the first variation of $J[y]$, denoted by $J_1[h]$.

So a necessary condition for $y(t)$ to be a local minimum is

$$J_1[h] = \int_{\rho^2(a)}^{\rho(b)} \{L_{y^\rho}(t, y^\rho(t), y^{\nabla}(t))h^\rho(t) + L_{y^\nabla}(t, y^\rho(t), y^{\nabla}(t))h^{\nabla}(t)\}\nabla t = 0$$

for all $h \in C^1[\rho^2(a), \rho(b)]$ with $h(\rho^2(a)) = h(\rho(b)) = 0$.

Using the properties of ∇-integral, we have

$$\begin{aligned}
J_1[h] &= \int_{\rho(a)}^{\rho(b)} \{L_{y^\rho}(t, y^\rho, y^\nabla)h^\rho + L_{y^\nabla}(t, y^\rho, y^\nabla)h^\nabla\}\nabla t \\
&\quad + \int_{\rho^2(a)}^{\rho(a)} \{L_{y^\rho}(t, y^\rho, y^\nabla)h^\rho + L_{y^\nabla}(t, y^\rho, y^\nabla)h^\nabla\}\nabla t \\
&= \int_{\rho(a)}^{\rho(b)} \{L_{y^\rho}(t, y^\rho, y^\nabla)h^\rho + L_{y^\nabla}(t, y^\rho, y^\nabla)h^\nabla\}\nabla t \\
&\quad + (\rho(a) - \rho^2(a))\{L_{y^\rho}(\rho(a), y(\rho^2(a)), y^\nabla(\rho(a)))h(\rho^2(a)) \\
&\quad + L_{y^\nabla}(\rho(a), y(\rho^2(a)), y^\nabla(\rho(a)))h^\nabla(\rho(a))\} \\
&= \int_{\rho(a)}^{\rho(b)} \{L_{y^\rho}(t, y^\rho, y^\nabla)h^\rho + L_{y^\nabla}(t, y^\rho, y^\nabla)h^\nabla\}\nabla t \\
&\quad + (\rho(a) - \rho^2(a))L_{y^\nabla}(\rho(a), y(\rho^2(a)), y^\nabla(\rho(a)))h^\nabla(\rho(a))
\end{aligned}$$

and, using the equality $(\rho(a) - \rho^2(a))h^\nabla(\rho(a)) = h(\rho(a)) - h(\rho^2(a))$, we obtain

$$\begin{aligned}
J_1[h] &= \int_{\rho(a)}^{\rho(b)} \{L_{y^\rho}(t, y^\rho, y^\nabla)h^\rho + L_{y^\nabla}(t, y^\rho, y^\nabla)h^\nabla\}\nabla t \\
&\quad + L_{y^\nabla}(\rho(a), y(\rho^2(a)), y^\nabla(\rho(a)))h(\rho(a)).
\end{aligned}$$

Integration by parts [12] gives

$$\begin{aligned}
&= \int_{\rho(a)}^{\rho(b)} L_{y^\rho}(t, y^\rho, y^\nabla)h^\rho\nabla t + (L_{y^\nabla}h)(\rho(b)) - (L_{y^\nabla}h)(\rho(a)) \\
&\quad - \int_{\rho(a)}^{\rho(b)} L_{y^\nabla}^\nabla(t, y^\rho, y^\nabla)h^\rho\nabla t + (L_{y^\nabla}h)(\rho(a)) \\
&= \int_{\rho(a)}^{\rho(b)} \{L_{y^\rho}(t, y^\rho, y^\nabla)h^\rho - L_{y^\nabla}^\nabla(t, y^\rho, y^\nabla)h^\rho\}\nabla t.
\end{aligned}$$

Again, using the property of ∇-integral,

$$\int_{\rho(b)}^{b} \{L_{y^\rho}(t, y^\rho, y^\nabla) - L_{y^\nabla}^\nabla(t, y^\rho, y^\nabla)\}h^\rho\nabla t = (\rho(b) - b)(\{L_{y^\rho} - L_{y^\nabla}^\nabla\}h)(\rho(b)) = 0,$$

we get

$$\int_{\rho(a)}^{b} \{L_{y^\rho}(t, y^\rho, y^\nabla) - L_{y^\nabla}^\nabla(t, y^\rho, y^\nabla)\}h^\rho\nabla t = 0 \tag{3.1}$$

for all $h \in C^1[\rho^2(a), \rho(b)]$ with $h(\rho^2(a)) = h(\rho(b)) = 0$.

Theorem 3.1. *If a function $y(t)$ provides a local extremum to the functional*

$$J[y] = \int_{\rho^2(a)}^{\rho(b)} L(t, y(\rho(t)), y^\nabla(t))\nabla t$$

where $y \in C^2[\rho^2(a), \rho(b)]$ and $y(\rho^2(a)) = A$, $y(\rho(b)) = B$, then y must satisfy the Euler–Lagrange equation

$$L_{y^\rho}(t, y^\rho, y^\nabla) - L_{y^\nabla}^\nabla(t, y^\rho, y^\nabla) = 0,$$

for $t \in [\rho(a), b]$.

Proof. The necessary condition for $J[y]$ to have an extremum for $y = y(t)$ is that

$$\varphi'(0) = \int_{\rho(a)}^{b} \{L_{y^\rho}(t, y^\rho, y^\nabla) - L^\nabla_{y^\nabla}(t, y^\rho, y^\nabla)\} h^\rho \nabla t = 0 \tag{3.2}$$

for all admissible h. Lemma 2.1 and the equality (3.1) imply that

$$L_{y^\rho}(t, y^\rho, y^\nabla) - L^\nabla_{y^\nabla}(t, y^\rho, y^\nabla) = 0 \tag{3.3}$$

a result known as a Euler–Lagrange equation. □

Now we state the theorem for the functional with several variables.

Theorem 3.2. *A necessary condition for the curve $y_i = y_i(t)$ for $i = 1, 2, \ldots, n$, to be extremal of the functional*

$$J[y_1, \ldots, y_n] = \int_{\rho^2(a)}^{\rho(b)} L(t, y_1^\rho, \ldots, y_n^\rho, y_1^\nabla, \ldots, y_n^\nabla) \nabla t$$

where $y_i \in C^2[\rho^2(a), \rho(b)]$ and $y_i(\rho^2(a)) = A$, $y_i(\rho(b)) = B$, is that the functions $y_i(t)$ satisfy the Euler–Lagrange equations

$$L_{y^\rho} - L^\nabla_{y^\nabla} = 0.$$

Next we are concerned with minimizing

$$J[y, u] = \int_{\rho^2(a)}^{\rho(b)} L(t, y^\rho(t), u^\rho(t)) \nabla t,$$

among all pairs (y, u) such that

$$y^\nabla(t) = f(t, y^\rho(t), u^\rho(t)),$$

together with appropriate conditions on endpoints. We note that the state equation may be considered as a pointwise constraint that can be treated by introducing a multiplier $p(t)$ such that $p^\rho(t)$ is a ∇-differentiable function on $[\rho^2(a), \rho(b)]_\kappa$. Therefore, we consider the functional

$$J^*[y, u, p, y^\nabla] = \int_{\rho^2(a)}^{\rho(b)} [L(t, y^\rho(t), u^\rho(t)) + p^\rho(t)(f(t, y^\rho(t), u^\rho(t)) - y^\nabla(t))] \nabla t.$$

The optimal solutions for our initial variational problem should be a solution of the Euler–Lagrange equation for J^* regarded as a function of the four variables (y, u, p, y^∇). If we put

$$G(t, u^\rho, p^\rho, y^\rho, u^\nabla, p^\nabla, y^\nabla) = L(t, y^\rho, u^\rho) + p^\rho(f(t, y^\rho, u^\rho) - y^\nabla),$$

then the Euler–Lagrange system can be written

$$G_{y^\rho} = G^\nabla_{y^\nabla}, \quad G_{u^\rho} = G^\nabla_{u^\nabla}, \quad G_{p^\rho} = G^\nabla_{p^\nabla},$$

that is

$$\begin{aligned} &L_{y^\rho}(t, y^\rho, u^\rho) + p^\rho f_{y^\rho}(t, y^\rho, u^\rho) + (p^\rho)^\nabla = 0, \\ &L_{u^\rho}(t, y^\rho, u^\rho) + p^\rho f_{u^\rho}(t, y^\rho, u^\rho) = 0, \\ &f(t, y^\rho, u^\rho) - y^\nabla = 0. \end{aligned}$$

Theorem 3.3. *Let f be linear in (y^ρ, u^ρ) and L concave in (y^ρ, u^ρ) for each fixed t. Then every solution of the system of optimality with the appropriate endpoint conditions (including transversality) will be an optimal solution of the variational problem.*

Proof. Assume that the pair (y, u) satisfies all the optimality conditions, and let $(\tilde{y}, \tilde{u})$ be any other admissible pair. We will measure the difference

$$J[\tilde{y}, \tilde{u}] - J[y, u]$$

and conclude that it cannot be positive. This implies that (y, u) is indeed optimal.

Due to the hypotheses of linearity and concavity assumed in the statement of the theorem, we can write

$$\begin{aligned}
J[\tilde{y}, \tilde{u}] - J[y, u] &= \int_{\rho^2(a)}^{\rho(b)} |L(t, \tilde{y}^\rho, \tilde{u}^\rho) - L(t, y^\rho, u^\rho)|\nabla t \\
&\le \int_{\rho^2(a)}^{\rho(b)} \{L_{y^\rho}(t, y^\rho, u^\rho)(\tilde{y}^\rho - y^\rho) + L_{u^\rho}(t, y^\rho, u^\rho)(\tilde{u}^\rho - u^\rho)\}\nabla t \\
&= \int_{\rho^2(a)}^{\rho(b)} \{(-p^\rho f_{y^\rho}(t, y^\rho, u^\rho) - (p^\rho)^\nabla)(\tilde{y}^\rho - y^\rho) - p^\rho f_{u^\rho}(t, y^\rho, u^\rho)(\tilde{u}^\rho - u^\rho)\}\nabla t \\
&= -\int_{\rho^2(a)}^{\rho(b)} p^\rho\{(y^\nabla - \tilde{y}^\nabla) + f_{y^\rho}(t, y^\rho, u^\rho)(\tilde{y}^\rho - y^\rho) + f_{u^\rho}(t, y^\rho, u^\rho)(\tilde{u}^\rho - u^\rho)\}\nabla t \\
&= -\int_{\rho^2(a)}^{\rho(b)} p^\rho\{f(t, y^\rho, u^\rho) - f(t, \tilde{y}^\rho, \tilde{u}^\rho) + f_{y^\rho}(t, y^\rho, u^\rho)(\tilde{y}^\rho - y^\rho) \\
&\quad + f_{u^\rho}(t, y^\rho, u^\rho)(\tilde{u}^\rho - u^\rho)\}\nabla t \\
&= 0. \quad \square
\end{aligned}$$

4. A model in economics

Most dynamic optimization problems in economics are set up in the following form: a consumer is seeking to maximize his lifetime utility subject to certain constraints. During each period in his life a consumer has to make a decision concerning how much to consume and how much to spend. If the consumer consumes more today, the "punishment" comes in the form of foregone consumption tomorrow. To be more precise, the punishment is not just the consumption itself but the *utility* we derive from consumption. So, it is an optimal control model — the solution is a function that describes optimal behavior for an individual. The solution shows how much one should consume each period to insure that one achieves maximum lifetime consumption.

We will start by describing the basic intuition behind the problem, explaining why we choose particular functions and constraints and developing some terminology.

Utility is the value function of the consumer that one wants to maximize. Utility can depend on numerous variables, but it typically depends on consumption. In this simple example, utility depends only on consumption of some generic product C. Utility $u(C)$, or the satisfaction we derive from consumption, has $u'(C) > 0$ and $u''(C) < 0$. This means that consumers always would like to consume more (because each additional unit generates positive utility) but each additional unit consumed during the same period generates less utility than the previous unit consumed within the same period. We call this property of utility function the Law of Diminishing Marginal Utility.

Discrete model:

What makes this a dynamic problem is that a consumer has to make decisions not just about one period but about the sequence of C's: $C_0, C_1, \ldots, C_T$, where T can be a finite number or ∞ (a consumer is a family that lives forever). Because we have a limited amount of resources, there is a trade-off between consuming today and consuming tomorrow. So the problem is to find a consumption path that would maximize *lifetime utility* U:

$$U = \sum_{s=0}^{T} \left(\frac{1}{1+\delta}\right)^s u(C_s),$$

where C_s is consumption during period s, u is one-period utility, and U is the lifetime utility. There is also the parameter $\delta \in (0, 1)$, which is the discount factor — we prefer to consume today rather than tomorrow. In other words, we value future consumption less than current consumption, so we discount the future at the rate δ. Loosely speaking,

δ is our internal interest rate, which reflects how much we are willing to give up today to increase consumption tomorrow.

So, the problem can be expressed as the value function to be maximized subject to a certain constraint:

$$\max U = \sum_{s=0}^{T} \left(\frac{1}{1+\delta}\right)^{s} u(C_s)$$

subject to $A_{s+1} = (1+r)A_s + Y_s - C_s$, for all $s \in [0, T)$ and $A_T \left(\frac{1}{1+r}\right)^T \geq 0$.

In other words, we have to maximize our lifetime utility, but we are constrained by the fact that the value of our consumption must be equal to the value of our income plus the assets that we might have. In addition to that, we have another constraint $A_T \left(\frac{1}{1+r}\right)^T \geq 0$ that can be interpreted as "we are not allowed to borrow without limit:" The present value of the last period asset holding has to be nonnegative (the optimal level is, naturally zero — we want to spend all the money we've got, and we don't care about leaving money behind after we die; so, using our intuition, we could make it an equality constraint).

Continuous model:

Following similar reasoning as above, we consider the problem of maximizing the lifetime utility, which is the sum of instantaneous utilities

$$U = \int_0^T u\left(C_s\right) e^{-\delta s} \mathrm{d}s$$

with respect to the path $\{C_s\}_{s=0}^T$, subject to constraint

$$A_s' = A_s r + Y_s - C_s.$$

Time scales model:

We maximize

$$U = \int_0^{\sigma(T)} u(C(\rho(s)))\hat{e}_{-\delta}(\rho(s), 0)\nabla s$$

subject to constraint

$$A^{\nabla}(s) = rA(\rho(s)) + Y(\rho(s)) - C(\rho(s)), \quad s \in [\sigma(0), T].$$

Note that this model includes the two above as special cases, namely $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$, respectively.

In this model:

$$G(s, x, y, z, w) = u(x)\hat{e}_{-\delta}(\rho(s), a) + \mu(\rho(s))[x - rz + w - Y(\rho(s))].$$

Without loss of generality, we can replace $\rho(s)$ by t and obtain

$$G(t, x, y, z, w) = u(x)\hat{e}_{-\delta}(t, 0) + \mu(t)[x - rz + w - Y(t)],$$

so Euler's equations become

$$u'(C)\hat{e}_{-\delta}(t, 0) + \mu(t) = 0,$$
$$-r\mu(t) - \mu^{\nabla}(t) = 0.$$

We note that these equations provide a unification of the Euler–Lagrange equations from both the discrete (constant time intervals) and continuous approaches. These equations also allow for more complicated applications than the discrete and continuous models allow. A consumer might have income from work or asset holdings arriving at unequal time intervals (say, from paychecks, dividend payments and rent payments arriving at different time intervals) and/or make expenditures at unequal time intervals. Such a problem could conceivably be studied by a constant time interval model or continuous model; however, the time scales approach allows for a more rigorous and probably more accurate solution. We continue analyzing this model as follows.

Substitution gives us the following dynamic equation:

$$-r[-u'(C(t))\hat{e}_{-\delta}(t,0)] - u'(C(t))\delta\hat{e}_{-\delta}(t,0) + \hat{e}_{-\delta}(\rho(t),0)[u'(C(t))]^{\nabla} = 0,$$

which reduces to

$$\hat{e}_{-\delta}(\rho(t),0)[u'(C(t))]^{\nabla} = [\delta - r]\hat{e}_{-\delta}(t,0)u'(C(t)).$$

Using a property of the nabla exponential function, namely $\hat{e}_{-\delta}(\rho(t),0) = (1+\delta\nu(t))\hat{e}_{-\delta}(t,0)$ [13, Theorem 3.15 (ii)], we have

$$[u'(C(t))]^{\nabla} = \frac{\delta - r}{1+\delta\nu(t)}u'(C(t)). \tag{4.1}$$

By use of the chain rule [3], we have

$$\frac{(u'(C(t)))^{\nabla}}{u'(C(t))} = \frac{\{\int_0^1 u''(C(t)+h\nu(t)C^{\nabla}(t))\mathrm{d}h\}C^{\nabla}(t)}{u'(C(t))} = \frac{\delta - r}{1+\delta\nu(t)}.$$

We can think of $\frac{[u'(C(t))]^{\nabla}}{u'(C(t))}$ as the growth rate of marginal utility. Next we obtain

$$\left(\frac{\delta - r}{1+\delta\nu(t)}\right)\frac{1}{C^{\nabla}(t)} < 0,$$

if the utility function is concave ($u' > 0, u'' < 0$). This last inequality states that the growth rate of consumption, $C^{\nabla}(t)$, is positive if $\delta < r$ and negative when $\delta > r$. Therefore, if $r > \delta$ (the market interest rate r is higher than internal rate of preference δ), the consumer will wait to consume until later periods. If $\delta > r$, the consumer is impatient and will consume more in the earlier periods and less in the future periods. For example, if $u(C) = \ln C$, we can express the left hand side of (4.1) in terms of the growth rate of consumption. So, given specific parameter values for δ and r and a jump function $\nu(t)$, we can find a dynamic equation for consumption on a time scale.

One possible economic interpretation of this equation is best understood by contrasting it with the discrete or continuous cases. In those two conventional setups, the growth rate of consumption is constant. When the model is solved using time scales, it shows that growth rates of consumption can fluctuate if consumption does not take place at fixed intervals (due to $\nu(t)$). Thus, the time scales model provides information for a problem for not evenly spaced intervals, for which the standard continuous and discrete models do not. When the discount rate is equal to the interest rate, $\delta - r = 0$ and the consumption level does not depend on the time scale.

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