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QUASILINEARIZATION FOR A NONLINEAR INITIAL VALUE PROBLEM ON TIME SCALES

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ABSTRACT. In the spotlight of this study is a particular type of first order dynamic initial value problem of the form

$$u^{\Delta} = f(t,u) + g(t,u), \quad u(t_0) = u_0,$$

where $f, g \in C_{rd}\left[\mathbb{T}^{\kappa} \times \mathbb{R}, \mathbb{R}\right]$ are nondecreasing and nonincreasing in u, respectively. A quasilinearization technique utilizing the nature of natural lower and upper solutions as well as coupled lower and upper solutions is developed for this problem. Beginning with the existence of coupled lower and upper solutions, the goal is to create two sequences of solutions, one that converges to a minimal solution and one that converges to a maximal solution.

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1. INTRODUCTION

The method of upper and lower solutions has been effectively used for proving the existence results for a wide variety of nonlinear problems. When coupled with the monotone iterative technique one obtains a constructive procedure for obtaining the solutions of the nonlinear problems besides enabling the study of the qualitative properties of the solutions. A very comprehensive introduction to the monotone iterative techniques is given in [11].

This method has further been exploited in combination with the method of quasilinearization, to obtain concurrently the lower and upper bounding monotone sequences, whose elements are solutions of linear problems, and hence are easier to obtain, which converge quadratically to the solution. This technique, known as the *generalized quasilinearization* has also been effectively used to study nonlinear problems and developed further in [13].

In the context of "time scales", in [8] Kaymakçalan had an initial attempt to developing the method of lower and upper solutions for obtaining extremal solutions of dynamic initial value problems. Further in [9] several other contributions in the

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direction of monotone methods and quasilinerization on time scales have been included, and finally in [7] Eloe has developed the method of quasilinearization for dynamic equations on compact measure chains, pioneering to several other contributions ([1, 2, 4]) in the area.

In the recent work of Akın-Bohner and Bohner [3] and in [6, Chapter 2], one of the forms of the generalized *logistic* dynamic equation (or *Verhulst* equation) appears as

$$x^{\Delta} = (p - f x^{\sigma})\, x, \tag{1.1}$$

where $p \in \mathcal{R}$ and $f \in C_{rd}$, with $\mathcal{R}$ denoting the *regressive group* as given in [5]. For further details of the notions related to the time scales calculus we refer to [5, 6]. Now although in the case of the above specific nonlinear equation, (1.1), Akın-Bohner and Bohner illustrate how a solution v of the corresponding linear equation

$$v^{\Delta} = -p(t)v^{\sigma} + f(t) \tag{1.2}$$

can be utilized to give a closed form solution of the nonlinear equation (1.1), (see [6, Theorem 2.24]), one may not always be so lucky. So the main subject of this paper, namely using quasilinearization technique to construct sequences converging first to extremal solutions and then further to the unique solution of the nonlinear IVP proves to be especially useful in such cases when corresponding linear equations and their solutions are not easily available.

The above considered *logistic equation* model (1.1) motivates one to consider initial value problems where the right-hand side consists of the difference of two functions $f(t,u)$ and $g(t,u)$ which are monotone in u. Also attention is focused on coupled lower and upper solutions instead of the usual separate lower and upper solutions. Our main aim in this work is to construct monotone, bounded sequences which converge uniformly to the minimal and maximal solutions of the problem, and it will be a subject of a future project to extend the results obtained in this work to the case of developing the convergence of the constructed monotone sequences to the unique solution of the problem. In the case when a closed form solution of the original nonlinear problem is known, such as in Theorem 2.24 of [6], exact description of the form of the sequences of lower and upper solutions obtained with the quasilinearization technique will be formulated, so as to coincide with the known solution in the limiting case.

2. PRELIMINARY CONCEPTS AND RESULTS

Our goal is to gain insight about lower and upper solutions of the dynamic IVP

$$u^{\Delta} = f(t,u) + g(t,u) \tag{2.1}$$

$$u(t_0) = u_0 \tag{2.2}$$

in the particular case when $f, g \in C_{rd}\left[\mathbb{T}^\kappa \times \mathbb{R}, \mathbb{R}\right]$ are respectively nondecreasing and nonincreasing in u on a time scale $\mathbb{T}^\kappa$. Let us begin with the usual definition of upper and lower solutions.

Definition 2.1. A function $\alpha \in C_{rd}\left[\mathbb{T}^\kappa, \mathbb{R}\right]$ is said to be a lower solution of

$$u^\Delta = h(t, u) \tag{2.3}$$

if

$$\alpha^\Delta \leq h(t, \alpha)$$

and, similarly, β is said to be an upper solution of (2.3) if

$$\beta^\Delta \geq h(t, \beta).$$

The concept of lower and upper solutions can be generalized to include coupled lower and upper solutions. Utilizing these generalized notions given by Lakshmikantham and Köksal in [12], and further employed by Lawrence and Kaymakçalan for dynamic initial value problems on time scales in [10], we define natural lower and upper solutions as well as the three types of coupled lower and upper solutions for our problem.

Definition 2.2. Let α and β be rd-continuously differentiable functions such that $\alpha^\sigma(t) \leq \beta(t)$ on $\mathbb{T}^\kappa$. Then α and β are

i) Natural lower and upper solutions of (2.1)–(2.2) if

$$\begin{aligned} \alpha^\Delta &\leq f\left(t, \alpha^\sigma\right) + g\left(t, \alpha^\sigma\right), & \alpha\left(t_0\right) \leq u_0, \\ \beta^\Delta &\geq f\left(t, \beta^\sigma\right) + g\left(t, \beta^\sigma\right), & \beta\left(t_0\right) \geq u_0; \end{aligned}$$

ii) Coupled lower and upper solutions of Type I of (2.1)–(2.2) if

$$\begin{aligned} \alpha^\Delta &\leq f\left(t, \alpha^\sigma\right) + g\left(t, \beta^\sigma\right), & \alpha\left(t_0\right) \leq u_0, \\ \beta^\Delta &\geq f\left(t, \beta^\sigma\right) + g\left(t, \alpha^\sigma\right), & \beta\left(t_0\right) \geq u_0; \end{aligned}$$

iii) Coupled lower and upper solutions of Type II of (2.1)–(2.2) if

$$\begin{aligned} \alpha^\Delta &\leq f\left(t, \beta^\sigma\right) + g\left(t, \alpha^\sigma\right), & \alpha\left(t_0\right) \leq u_0, \\ \beta^\Delta &\geq f\left(t, \alpha^\sigma\right) + g\left(t, \beta^\sigma\right), & \beta\left(t_0\right) \geq u_0; \end{aligned}$$

iv) Coupled lower and upper solutions of Type III of (2.1)–(2.2) if

$$\begin{aligned} \alpha^\Delta &\leq f\left(t, \beta^\sigma\right) + g\left(t, \beta^\sigma\right), & \alpha\left(t_0\right) \leq u_0, \\ \beta^\Delta &\geq f\left(t, \alpha^\sigma\right) + g\left(t, \alpha^\sigma\right), & \beta\left(t_0\right) \geq u_0. \end{aligned}$$

Vast amount of qualitative and quantitative results for various types of differential equations involving the above four cases have been developed (see [12]). In fact, the nature of the functions f and g reduce the number of distinct cases to two. Namely, natural lower and upper solutions satisfy the inequalities that define Type II coupled lower and upper solutions, that is,

$$\begin{aligned} \alpha^{\Delta} &\leq f(t,\alpha^{\sigma}) + g(t,\alpha^{\sigma}) \leq f(t,\beta^{\sigma}) + g(t,\alpha^{\sigma}), \\ \beta^{\Delta} &\geq f(t,\beta^{\sigma}) + g(t,\beta^{\sigma}) \geq f(t,\alpha^{\sigma}) + g(t,\beta^{\sigma}). \end{aligned}$$

In a similar manner, Type III coupled lower and upper solutions also satisfy the inequalities that define Type II coupled lower and upper solutions. This observation is verified by the following inequalites:

$$\begin{aligned} \alpha^{\Delta} &\leq f(t,\beta^{\sigma}) + g(t,\beta^{\sigma}) \leq f(t,\beta^{\sigma}) + g(t,\alpha^{\sigma}), \\ \beta^{\Delta} &\geq f(t,\alpha^{\sigma}) + g(t,\alpha^{\sigma}) \geq f(t,\alpha^{\sigma}) + g(t,\beta^{\sigma}). \end{aligned}$$

Therefore it will be enough to show that our results hold merely for Type I and Type II lower and upper coupled solutions. Each of these interesting cases will be addressed in the work that follows.

The construction of our sequences of lower and upper solutions which will then be shown to converge to the unique solution of the IVP (2.1)–(2.2), requires certain natural relationships between lower and upper solutions as well as the solutions of (2.1)–(2.2) to hold. These relationships have been verified by Kaymakçalan [8] in the case of the right hand side function of (2.1) being a general rd-continuos function, without necessarily having to split into a sum of nondecreasing and nonincresaing functions, and therefore similar results in the context of our problem are presented below without proof.

Theorem 2.3. *Let α and β be either Type I or Type II coupled lower and upper solutions of* (2.1)–(2.2)*, respectively. Assume $f(t,x)$ satisfies*

$$f(t,x) - f(t,y) \leq L(x-y) \quad \textit{for} \quad x \geq y \tag{2.4}$$

and $g(t,x)$ is nonincreasing in x. Then $\alpha(t_0) \leq \beta(t_0)$ implies that $\alpha(t) \leq \beta(t)$ for $t \in \mathbb{T}^{\kappa}$.

Under these assumptions, any solution of (2.1)–(2.2) is squeezed between lower and upper solutions (2.1)–(2.2) where lower-upper solutions are given in terms of Definition 2.1. This result is formally stated next.

Theorem 2.4. *Assume the conditions on f and g of Theorem 2.3 are satisfied. Then any solution $u(t)$ of* (2.1)–(2.2)*, satisfying*

$$\alpha(t_0) \leq u(t_0) \leq \beta(t_0),$$

with α and β respectively being lower and upper solutions of (2.1)–(2.2)*, also satisfies*

$$\alpha(t) \leq u(t) \leq \beta(t),$$

for $t \in \mathbb{T}^\kappa$.

Under the assumption that lower and upper solutions exist, we can verify the existence of a solution of (2.1)–(2.2) that is bounded above by the upper solution and below by the lower solution, for each $t \in \mathbb{T}^\kappa$. Kaymakçalan also established this result [8, Theorem1.3], again for a general problem of the form

$$u^\Delta = f(t,u), \quad u(t_0) = u_0$$

when $f \in C_{rd}\left[\mathbb{T}^\kappa \times \mathbb{R}, \mathbb{R}\right]$, and below we state the equivalent result in the context of our problem.

Theorem 2.5. *Let $\alpha, \beta \in C_{rd}\left[\mathbb{T}^\kappa, \mathbb{R}\right]$ be either Type I or Type II lower and upper solutions of* (2.1)–(2.2)*, respectively, such that $\alpha(t) \leq \beta(t)$ for $t \in \mathbb{T}^\kappa$, and $f, g \in C_{rd}\left[\Omega, \mathbb{R}\right]$ be bounded on Ω, where Ω is the closed set determined by $\alpha(t)$ and $\beta(t)$, namely*

$$\Omega = \left\{(t,u) : \ \alpha(t) \leq u \leq \beta(t), \ t \in \mathbb{T}^\kappa\right\}.$$

Then there exists a solution, $u(t)$, of (2.1)–(2.2) *such that $\alpha(t) \leq u(t) \leq \beta(t)$ on $\mathbb{T}^\kappa$ whenever $\alpha(t_0) \leq u(t_0) \leq \beta(t_0)$.*

3. THE MAIN RESULTS

The existence of two convergent sequences of solutions that converge one to a maximal one to a minimal solution of (2.1)–(2.2) is now established by way of construction. First we consider lower and upper coupled solutions of Type I.

Theorem 3.1. *Let f and g be functions with properties described in Theorem 2.5. In addition, assume that*

i) *f_x, f_{xx}, g_x, g_{xx} are continuous functions on $\mathbb{T}^\kappa \times \mathbb{R}$;*
ii) *$f_{xx} \geq 0$ and $g_{xx} \leq 0$ for $(t,x) \in \mathbb{T}^k \times \mathbb{R}$.*

Let α_0 and β_0 denote Type I coupled lower and upper solutions of (2.1)–(2.2) *respectively. Then there exist sequences $\{\alpha_n\}$, $\{\beta_n\}$ that converge in the space of continuous functions to minimal and maximal solutions, respectively, of* (2.1)–(2.2)*.*

Proof. The additional assumptions made on f and g yield the inequalities

$$\frac{f(t,x) - f(t,y)}{x-y} \geq f_x(t,y) \quad \text{and} \quad \frac{g(t,x) - g(t,y)}{x-y} \leq g_x(t,y).$$

For the construction of our sequences we need two linearization terms. First define a function $F : \mathbb{T}^\kappa \times \mathbb{R} \to \mathbb{R}$ by

$$F(t, x^\sigma; \alpha_0, \beta_0) = f(t, \alpha_0^\sigma) + f_x(t, \alpha_0^\sigma)(x^\sigma - \alpha_0^\sigma) + g(t, \beta_0^\sigma) + g_x(t, \alpha_0^\sigma)(x^\sigma - \alpha_0^\sigma),$$

where α_0 and β_0 are respectively Type I lower and upper solutions of (2.1)–(2.2). Utilizing this function we create the linear dynamic equation

$$x^{\Delta} = F\left(t, x^{\sigma}; \alpha_0, \beta_0\right) \tag{3.1}$$

satisfying initial condition (2.2). Using the natures of α_0 and β_0, and the assumptions on f and g, we can establish th inequalities

$$\begin{aligned}
\alpha_0^{\Delta} &\leq f\left(t, \alpha_0^{\sigma}\right) + g\left(t, \beta_0^{\sigma}\right) \\
&\leq f\left(t, \alpha_0^{\sigma}\right) + f_x\left(t, \alpha_0^{\sigma}\right)\left(\alpha_0^{\sigma} - \alpha_0^{\sigma}\right) + g\left(t, \beta_0^{\sigma}\right) + g_x\left(t, \alpha_0^{\sigma}\right)\left(\alpha_0^{\sigma} - \alpha_0^{\sigma}\right) \\
&= F\left(t, \alpha_0^{\sigma}; \alpha_0, \beta_0\right)
\end{aligned}$$

and

$$\begin{aligned}
\beta_0^{\Delta} &\geq f\left(t, \beta_0^{\sigma}\right) + g\left(t, \alpha_0^{\sigma}\right) \\
&\geq f\left(t, \alpha_0^{\sigma}\right) + f_x\left(t, \alpha_0^{\sigma}\right)\left(\beta_0^{\sigma} - \alpha_0^{\sigma}\right) + g\left(t, \beta_0^{\sigma}\right) - g_x\left(t, \alpha_0^{\sigma}\right)\left(\beta_0^{\sigma} - \alpha_0^{\sigma}\right) \\
&\geq f\left(t, \alpha_0^{\sigma}\right) + f_x\left(t, \alpha_0^{\sigma}\right)\left(\beta_0^{\sigma} - \alpha_0^{\sigma}\right) + g\left(t, \beta_0^{\sigma}\right) + g_x\left(t, \alpha_0^{\sigma}\right)\left(\beta_0^{\sigma} - \alpha_0^{\sigma}\right) \\
&= F\left(t, \beta_0^{\sigma}; \alpha_0, \beta_0\right).
\end{aligned}$$

Therefore, α_0 and β_0 are Type I lower and upper solutions of (3.1)–(2.2). In view of the work of Kaymakçalan [8] and the above Theorem 2.4 and Theorem 2.5, we know that there exists a solution of (3.1)–(2.2), call it α_1, such that

$$\alpha_0\left(t\right) \leq \alpha_1\left(t\right) \leq \beta_0\left(t\right)$$

for $t \in \mathbb{T}^{\kappa}$.

A second linearization is defined using the function $G : \mathbb{T}^{\kappa} \times \mathbb{R} \rightarrow \mathbb{R}$, according to

$$G\left(t, x^{\sigma}; \alpha_0, \beta_0\right) = f\left(t, \beta_0^{\sigma}\right) + f_x\left(t, \alpha_0^{\sigma}\right)\left(x^{\sigma} - \beta_0^{\sigma}\right) + g\left(t, \alpha_0^{\sigma}\right) + g_x\left(t, \beta_0^{\sigma}\right)\left(x^{\sigma} - \beta_0^{\sigma}\right),$$

yielding the dynamic equation

$$x^{\Delta} = G\left(t, x^{\sigma}; \alpha_0, \beta_0\right), \tag{3.2}$$

satisfying initial condition (2.2). Again we can verify that α_0 and β_0 are respectively Type I lower and upper solutions of (3.2)–(2.2) by working our way through the following inequalities,

$$\begin{aligned}
\alpha_0^{\Delta} &\leq f\left(t, \alpha_0^{\sigma}\right) + g\left(t, \beta_0^{\sigma}\right) \\
&\leq f\left(t, \beta_0^{\sigma}\right) - f_x\left(t, \alpha_0^{\sigma}\right)\left(\beta_0^{\sigma} - \alpha_0^{\sigma}\right) + g\left(t, \alpha_0^{\sigma}\right) + g_x\left(t, \alpha_0^{\sigma}\right)\left(\beta_0^{\sigma} - \alpha_0^{\sigma}\right) \\
&\leq f\left(t, \beta_0^{\sigma}\right) + f_x\left(t, \alpha_0^{\sigma}\right)\left(\alpha_0^{\sigma} - \beta_0^{\sigma}\right) + g\left(t, \alpha_0^{\sigma}\right) + g_x\left(t, \alpha_0^{\sigma}\right)\left(\alpha_0^{\sigma} - \beta_0^{\sigma}\right) \\
&\leq f\left(t, \beta_0^{\sigma}\right) + f_x\left(t, \alpha_0^{\sigma}\right)\left(\alpha_0^{\sigma} - \beta_0^{\sigma}\right) + g\left(t, \alpha_0^{\sigma}\right) + g_x\left(t, \beta_0^{\sigma}\right)\left(\alpha_0^{\sigma} - \beta_0^{\sigma}\right) \\
&= G\left(t, \alpha_0^{\sigma}; \alpha_0, \beta_0\right)
\end{aligned}$$

and

$$\begin{aligned}
\beta_0^{\Delta} &\geq f\left(t, \beta_0^{\sigma}\right) + g\left(t, \alpha_0^{\sigma}\right) \\
&\geq f\left(t, \beta_0^{\sigma}\right) + f_x\left(t, \alpha_0^{\sigma}\right)\left(\beta_0^{\sigma} - \beta_0^{\sigma}\right) + g\left(t, \alpha_0^{\sigma}\right) + g_x\left(t, \beta_0^{\sigma}\right)\left(\beta_0^{\sigma} - \beta_0^{\sigma}\right) \\
&= G\left(t, \alpha_0^{\sigma}; \alpha_0, \beta_0\right).
\end{aligned}$$

Note that the previously mentioned result of Kaymakçalan [8] comparing lower and upper solutions, presented in in the context of our problem as given by Theorem 2.3 is employed also in arriving at the above inequalities. Existence theorem, Theorem 2.5 again yields a solution of (3.2)–(2.2), name it β_1, such that

$$\alpha_0\left(t\right) \leq \beta_1\left(t\right) \leq \beta_0\left(t\right)$$

for $t \in \mathbb{T}^{\kappa}$.

Taking into consideration the the assumptions on α_1 and β_1 and working through the following inequalities, we arrive at the conclusion that α_1 and β_1 are Type I lower and upper solutions of (2.1)–(2.2) as well. Let us verify this claim:

$$\begin{aligned}
\alpha_1^{\Delta} &= f\left(t, \alpha_0^{\sigma}\right) + f_x\left(t, \alpha_0^{\sigma}\right)\left(\alpha_1^{\sigma} - \alpha_0^{\sigma}\right) + g\left(t, \beta_0^{\sigma}\right) + g_x\left(t, \alpha_0^{\sigma}\right)\left(\alpha_1^{\sigma} - \alpha_0^{\sigma}\right) \\
&\leq f\left(t, \alpha_1^{\sigma}\right) + g\left(t, \beta_0^{\sigma}\right) \\
&\leq f\left(t, \alpha_1^{\sigma}\right) + g\left(t, \beta_1^{\sigma}\right)
\end{aligned}$$

and

$$\begin{aligned}
\beta_1^{\Delta} &= f\left(t, \beta_0^{\sigma}\right) + f_x\left(t, \alpha_0^{\sigma}\right)\left(\beta_1^{\sigma} - \beta_0^{\sigma}\right) + g\left(t, \alpha_0^{\sigma}\right) + g_x\left(t, \beta_0^{\sigma}\right)\left(\beta_1^{\sigma} - \beta_0^{\sigma}\right) \\
&\geq f\left(t, \beta_0^{\sigma}\right) - f_x\left(t, \alpha_0^{\sigma}\right)\left(\beta_0^{\sigma} - \beta_1^{\sigma}\right) + g\left(t, \alpha_1^{\sigma}\right) \\
&\geq f\left(t, \beta_0^{\sigma}\right) - f_x\left(t, \beta_1^{\sigma}\right)\left(\beta_0^{\sigma} - \beta_1^{\sigma}\right) + g\left(t, \alpha_1^{\sigma}\right) \\
&\geq f\left(t, \beta_1^{\sigma}\right) + g\left(t, \alpha_1^{\sigma}\right).
\end{aligned}$$

Now, Theorem 2.3 yields $\alpha_1 \leq \beta_1$ and therefore we have established the string of inequalities

$$\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0.$$

Using the standard induction principle it can be shown that

$$\alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n,$$

where α_{n+1} satisfies

$$x^{\Delta} = F\left(t, x^{\sigma}; \alpha_n, \beta_n\right),$$

and β_{n+1} satisfies

$$x^{\Delta} = G\left(t, x^{\sigma}; \alpha_n, \beta_n\right).$$

Through this construction, two monotone and bounded sequences, $\{\alpha_n\}$, $\{\beta_n\}$ are formed and moreover, for each n, α_n is a lower solution and respectively, β_n is an upper solution of (2.1)–(2.2).

Since the set Ω, given in the hypotheses of Theorem 2.5, is compact, employing standard uniform convergence arguments, the limits,

$$\alpha = \lim_{n\to\infty} \alpha_n \quad \text{and} \quad \beta = \lim_{n\to\infty} \beta_n$$

are obtained. Furthermore, it can be shown that α and β are minimal and maximal solutions, respectively, of our original problem, (2.1)–(2.2) thus verifying the assertion of the theorem. $\square$

Remarks:

1) Theorem 3.1 actually serves two purposes:

i) It establishes the existence of two sequences of solutions, one that converges to a minimal solution of our original problem, and one that converges to a maximal solution.

ii) It provides a constructive technique for creating these sequences.

2) As given by [10, Corollary 3.1], in the case when the right-hand side term, $k(t,u) = f(t,u) + g(t,u)$, of (2.1) satisfies a Lipschitz condition in u, it can be shown (see [10]) that the IVP (2.1)–(2.2) has a unique solution $u(t)$ and moreover the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ constructed in the above Theorem 3.1 converge to this unique solution, thereby implying $\alpha(t) = u(t) = \beta(t)$.

Recall that the above result, Theorem 3.1, is only verified for Type I coupled lower and upper solutions. Next we focus on coupled solutions of Type II, thereby, in view of the observations following the Definition 2.2, establishing the existence of convergent sequences to the maximal and minimal solutions of (2.1)–(2.2) not only for Type II, but for Type III, and the usual natural upper and lower solutions as well.

Theorem 3.2. *Assume the conditions on f and g required in Theorem 3.1 are satisfied. Further assume the existence of α_0 and β_0, Type II coupled lower and upper solutions of* (2.1)–(2.2). *Then there exist convergent sequences α_n and β_n with*

$$\alpha = \lim_{n\to\infty} \alpha_n \quad \textit{and} \quad \beta = \lim_{n\to\infty} \beta_n$$

where α is a minimal solution of (2.1)–(2.2) *and β is, respectively, a maximal solution.*

Proof. The proof of Theorem 3.2 follows the same format as that of Theorem 3.1. The difference, of course, is in the definitions of the linearization terms necessary for the quasilinearization process. In the current situation, define

$$\begin{aligned} F(t, x^\sigma; \alpha_0, \beta_0) &= f(t, \beta_0^\sigma) + f_x(t, \beta_0^\sigma)(\alpha_0^\sigma - x^\sigma) \\ &\quad + g(t, \alpha_0^\sigma) + g_x(t, \beta_0^\sigma)(x^\sigma - \alpha_0^\sigma), \end{aligned}$$

and

$$G(t, x^\sigma; \alpha_0, \beta_0) = f(t, \alpha_0^\sigma) + f_x(t, \beta_0^\sigma)(\beta_0^\sigma - x^\sigma) + g(t, \beta_0^\sigma) + g_x(t, \beta_0^\sigma)(x^\sigma - \beta_0^\sigma).$$

Utilizing these definitions, we construct linear equations

$$x^\Delta = F(t, x^\sigma; \alpha_0, \beta_0) \tag{3.3}$$

and

$$x^\Delta = G(t, x^\sigma; \alpha_0, \beta_0) \tag{3.4}$$

and obtain, as before, corresponding solutions α_1, of (3.3), (2.2), and β_1, of (3.4), (2.2), such that

$$\alpha_0(t) \le \alpha_1(t) \le \beta_0(t)$$

and

$$\alpha_0(t) \le \beta_1(t) \le \beta_0(t).$$

As in the previous case, α_1 and β_1 can be shown to be lower and upper solutions, respectively, of (2.1)–(2.2) and hence employing Theorem 2.3 yields

$$\alpha_1(t) \le \beta_1(t).$$

Just as in the proof of Theorem 3.1, following similar induction and uniform convergence arguments, two convergent sequences $\{\alpha_n\}$ and $\{\beta_n\}$ whose limits are minimal and maximal solutions, respectively, of the original problem are obtained. □

Note that each of the four possible couplings of lower and upper solutions, utilizing α and β as given in Definition 2.2 is addressed in either Theorem 3.1 or Theorem 3.2, thus generalizing the results given for the usual concept of the lower and upper solution as indicated by Definition 2.1.

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