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THE EULER–MINDING ANALOGON AND OTHER IDENTITIES FOR CONTINUED FRACTIONS IN BANACH SPACES

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ABSTRACT. The Euler–Minding formulae and other identities for the numerators and denominators of the approximants of noncommutative continued fractions in Banach spaces are proved.

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1. INTRODUCTION

We investigate expressions of the form

$$B_0 + A_1(B_1 + A_2(\dots)^{-1})^{-1}, \tag{1.1}$$

where A_n, B_n are elements of a complex Banach algebra M, called noncommutative continued fraction (ncf). Wynn [4] developed a number of identities for such expressions. One of his results he announced as Euler–Minding analogon, in fact the scalar result is due to Stern [1].

In this paper we generalize the genuine Euler–Minding theorem for ncf and develop formulae for generating a ncf to given sequences of nth numerators and denominators. Furtheron we compute transformations, which are effictive in convergence theory of ncf.

2. DEFINITIONS AND NOTATIONS

Througout this paper M denotes a noncommutative Banach algebra with identity E and M^* the set of its invertible elements.

Definition 2.1. For $k \in \mathbb{N}$ let $S_k : N_k \to M$, $N_k \subseteq M$, $S_k(X) := A_k(B_k + X)^{-1}$, where $A_k, B_k \in M$. If $T_n := S_1 \circ \ldots \circ S_n(0)$ exists, we call $R_n := B_0 + T_n$ the nth approximant of (1.1).

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If R_n exists, it can be shown [4] that $R_n = P_n Q_n^{-1}$ (especially $Q_n \in M^*$), where P_n and Q_n are computed by the recurrence relations

$$\begin{aligned} &P_{-1} = E, \quad P_0 = B_0, \quad P_n = P_{n-1}B_n + P_{n-2}A_n \quad \text{for} \quad n \geq 1, \\ &Q_{-1} = 0, \quad Q_0 = E, \quad Q_n = Q_{n-1}B_n + Q_{n-2}A_n \quad \text{for} \quad n \geq 1. \end{aligned} \tag{2.1}$$

P_n and Q_n are called the nth numerator and denominator of (1.1), respectively.

Remark 2.2. To ensure the existence of R_n, $Q_n \in M^*$ is not sufficient, see [2]. Since this has no consequences on our investigations, we renounce to go into detail.

3. THE MAIN RESULTS

Wynn stated in [4] the identity

$$P_n Q_n^{-1} = B_0 + \sum_{k=0}^{n-1} (-1)^k A_1 B_1^{-1} Q_0 A_2 Q_2^{-1} Q_1 A_3 Q_3^{-1} \cdots Q_{k-1} A_{k+1} Q_{k+1}^{-1} \tag{3.1}$$

and announced it as the Euler–Minding formulae for ncf. In fact this is the generalization of Stern's result for ordinary continued fractions [1]. In the scalar case the Euler–Minding formulae enable us to compute both, the nth numerators and denominator of a continued fraction, without the recurrence relations (2.1). An analogous result for ncf is the following:

Theorem 3.1. *For the nth numerator P_n and denominator Q_n of* (1.1) *we have*

$$\begin{aligned} P_n = {} & B_0 B_1 \cdots B_n + \sum_{k=1}^{n} B_0 \cdots B_{k-2} A_k B_{k+1} \cdots B_n \\ & + \sum_{1 \leq k < i \leq n-1} B_0 \cdots B_{k-2} A_k B_{k+1} \cdots B_{i-1} A_{i+1} B_{i+2} \cdots B_n \\ & + \sum_{1 \leq k < i < j \leq n-1} B_0 \cdots B_{k-2} A_k B_{k+1} \cdots B_{i-1} A_{i+1} B_{i+2} \cdots B_{j-1} A_{j+1} B_{j+2} \cdots B_n + \ldots \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} Q_n = {} & B_1 \cdots B_n + \sum_{k=2}^{n} B_1 \cdots B_{k-2} A_k B_{k+1} \cdots B_n \\ & + \sum_{2 \leq k < i \leq n-1} B_1 \cdots B_{k-2} A_k B_{k+1} \cdots B_{i-1} A_{i+1} B_{i+2} \cdots B_n \\ & + \sum_{2 \leq k < i < j \leq n-1} B_1 \cdots B_{k-2} A_k B_{k+1} \cdots B_{i-1} A_{i+1} B_{i+2} \cdots B_{j-1} A_{j+1} B_{j+2} \cdots B_n + \ldots \end{aligned} \tag{3.3}$$

where $B_m \cdots B_d = E$, whenever $m > d$.

Remark 3.2. We have $[\frac{n+1}{2}]$ sums in (3.2) and (3.3).

Proof of 3.1. For the moment let $B_n \in M^*$, $n \in \mathbb{N}$. Define a sequence (U_n) as follows:

$$U_0 := B_0,\ U_1 := B_0B_1 + A_1 \quad \text{and} \quad U_n := U_{n-1}(B_n + B_{n-1}^{-1}A_n) \text{ for } n \geq 2.$$

Writing out U_n as an explicite sum, we split U_n up in two parts U_n' and U_n'', where U_n'' consists of all terms which contain at least one of the factors B_k^{-1}, hence $U_n = U_n' + U_n''$. We obtain $U_0 = U_0' = P_0$ and $U_1 = U_1' = P_1$ immediately by definition and

$$U_2 = U_1(B_2 + B_1^{-1}A_2) = U_1B_2 + B_0A_2 + A_1B_1^{-1}A_2 = U_1'B_2 + U_0'A_2 + U_2'',$$

hence $U_2' = U_1'B_2 + U_0'A_2 = P_2$. Further

$$\begin{aligned} U_n &= U_{n-1}B_n + U_{n-1}B_{n-1}^{-1}A_n = U_{n-1}B_n + (U_{n-2}B_{n-1} + U_{n-2}B_{n-2}^{-1}A_{n-1})B_{n-1}^{-1}A_n \\ &= (U_{n-1}' + U_{n-1}'')B_n + (U_{n-2}' + U_{n-2}'')A_n + U_{n-2}B_{n-2}^{-1}A_{n-1}B_{n-1}^{-1}A_n = U_n' + U_n''. \end{aligned}$$

For B_{n-1} does not appear in U_{n-2}, the term $U_{n-2}B_{n-2}^{-1}A_{n-1}B_{n-1}^{-1}$ is part of U_n'', thus $U_n' = U_{n-1}'B_n + U_{n-2}'A_n$, and we obtain by (2.1) that $P_n = U_n'$ for all $n \geq 0$. To compute U_n' we establish that $U_n = (B_0B_1 + A_1)(B_2 + B_1^{-1}A_2)\cdots(B_n + B_{n-1}^{-1}A_n)$ yields a part of U_n' if

(i) in two successive factors $(B_{k-1} + B_{k-2}^{-1}A_{k-1})$ and $(B_k + B_{k-1}^{-1}A_k)$ we multiply B_{k-1} by $B_{k-1}^{-1}A_k$; or
(ii) it is $B_0B_1\cdots B_n$ or $A_1B_2B_3\cdots B_n$.

Combining this we obtain (3.2). Then (3.3) follows immediately, for Q_n may be developed from P_{n-1} if we add 1 to the indices of all its elements. Finally we do not need any B_k^{-1} for computation of (3.2) and (3.3), so the formulae are granted for $B_k \in M$. □

For investigations of continued fractions, the continued fraction transformations play an important part. Equivalence transformation see [2, 3], contraction and extension see [3, 4] for ncf has already been studied. Another important transformation is the following. Let

$$E + A_1(E + A_2(\ldots)^{-1})^{-1} \tag{3.4}$$

be a ncf with its nth approximants R_n. We now want to create a ncf with the same approximants but in permuted succession, for example $R_1, R_0, R_3, R_2, \ldots$. This transformations are very interesting in convergence theory of ncf, because the transformed ncf converges exactly if (3.4) converges.

Theorem 3.3. *Assume* (3.4) *with nth numerators P_n and denominators Q_n, then the sequence of numerators respectively denominators of*

$$B_0' + A_1'(B_1' + A_2'(\ldots)^{-1})^{-1} \tag{3.5}$$

is:

(a) P_1, P_0, P_3, $P_2(E+A_3)$, P_5, $P_4(E+A_5), \ldots$, *respectively*, Q_1, Q_0, Q_3, $Q_2(E+A_3)$, Q_5, $Q_4(E+A_5), \ldots$ *if* $(E+A_{2n+1}) \in M^*$ *for all* $n \in \mathbb{N}$, *and we put*

$$B'_0 = E + A_1,\ B'_2 = A_2,\ B'_{2n} = (E + A_{2n-1})^{-1} A_{2n} \text{ for } n \geq 2,\ B'_{2n-1} = E \text{ for } n \geq 1$$

and $A'_1 = A_1$, $A'_3 = A_2 A_3$,

$$A'_{2n+1} = (E + A_{2n-1})^{-1} A_{2n} A_{2n+1} \text{ for } n \geq 2,\ A'_{2n} = E + A_{2n+1} \text{ for } n \geq 1,$$

(b) P_0, P_2, P_1, P_4, $P_3(E+A_4)$, P_6, $P_5(E+A_6), \ldots$, *respectively* Q_0, Q_2, Q_1, Q_4, $Q_3(E+A_4)$, Q_6, $Q_5(E+A_6), \ldots$, *if* $(E+A_{2n}) \in M^*$ *for all* $n \in \mathbb{N}$, *and we put*

$$B'_1 = E + A_2,\ B'_3 = A_3,\ B'_{2n+1} = (E + A_{2n})^{-1} A_{2n+1} \text{ for } n \geq 2,\ B'_{2n} = E \text{ for } n \geq 0$$

and

$$A'_1 = A_1,\ A'_{2n+1} = E + A_{2n+2} \text{ for } n \geq 1,$$
$$A'_2 = -A_2,\ A'_4 = A_3 A_4,\ A'_{2n} = (E + A_{2n-2})^{-1} A_{2n-1} A_{2n} \text{ for } n \geq 3.$$

Proof. (a) Denote P'_n and Q'_n as the numerators and denominators of (3.5). Then $P'_0 = P_1$, $P'_1 = P_0$, and by the recurrence relations (2.1) we obtain immediately $P'_2 = P_3$ and $P'_3 = P_2(E + A_2)$. Let $P'_{2n} = P_{2n+1}$ and $P'_{2n+1} = P_{2n}(E + A_{2n+1})$. Then (2.1) implies

$$\begin{aligned} P'_{2n+2} &= P'_{2n+1} B'_{2n+2} + P'_{2n} A'_{2n+2} \\ &= P_{2n}(E + A_{2n+1})(E + A_{2n+1})^{-1} A_{2n+2} + P_{2n+1}(E + A_{2n+2}) \\ &= P_{2n+1} + P_{2n} A_{2n+2} + P_{2n+1} A_{2n+2} = P_{2n+3} \end{aligned}$$

and

$$\begin{aligned} P'_{2n+3} &= P'_{2n+2} B'_{2n+3} + P'_{2n+1} A'_{2n+3} \\ &= P_{2n+3} + P_{2n}(E + A_{2n+1})(E + A_{2n+1})^{-1} A_{2n+2} A_{2n+3} \\ &= P_{2n+2} + P_{2n+1} A_{2n+3} + P_{2n} A_{2n+2} A_{2n+3} = P_{2n+2}(E + A_{2n+3}). \end{aligned}$$

Induction also implies the assertion for Q'_n. (b) follows analogously as (a). □

Finally we generate a ncf, whose numerators and denominators are given sequences.

Theorem 3.4. *Let* $(X_n)_{n=0}^{\infty} \subset M$, $(Y_n)_{n=0}^{\infty} \subset M^*$ *and* $Y_0 = E$. *If* $(X_n Y_n^{-1} - X_{n-1} Y_{n-1}^{-1}) \in M^*$ *for all* $n \in \mathbb{N}$, *then there exists a unique ncf* $B_0 + A_1(B_1 + A_2(\ldots)^{-1})^{-1}$ *with nth numerators* X_n *and denominators* Y_n *for all* $n \geq 0$. *For its partial numerators respectively denominators we obtain* $A_1 = X_1 - X_0 Y_1$,

$$A_n = Y_{n-2}^{-1}(X_{n-1} Y_{n-1}^{-1} - X_{n-2} Y_{n-2}^{-1})^{-1}(X_{n-1} Y_{n-1}^{-1} - X_n Y_n^{-1}) Y_n, \text{ for all } n \geq 2,$$

respectively $B_0 = X_0$, $B_1 = Y_1$,

$$B_n = Y_{n-1}^{-1}(X_{n-2} Y_{n-2}^{-1} - X_{n-1} Y_{n-1}^{-1})^{-1}(X_{n-2} Y_{n-2}^{-1} - X_n Y_n^{-1}) Y_n \text{ for all } n \geq 2.$$

Proof. Obviously we have $P_0 = X_0$, $P_1 = X_1$ and $Q_0 = Y_0$, $Q_1 = Y_1$. For $n \geq 2$ we construct A_n and B_n as follows. The recurrence relations (2.1) imply

$$X_n = X_{n-1}B_n + X_{n-2}A_n \quad \text{and} \quad Y_n = Y_{n-1}B_n + Y_{n-2}A_n. \tag{3.6}$$

Multiply the left equation in (3.6) by X_{n-1}^{-1} and the right equation in (3.6) by Y_{n-1}^{-1} to obtain by subtraction

$$Y_{n-1}^{-1}Y_n - X_{n-1}^{-1}X_n = (Y_{n-1}^{-1}Y_{n-2} - X_{n-1}^{-1}X_{n-2})A_n$$

and hence

$$Y_{n-2}^{-1}(X_{n-1}Y_{n-1}^{-1} - X_{n-2}Y_{n-2}^{-1})^{-1}(X_{n-1}Y_{n-1}^{-1} - X_nY_n^{-1})Y_n = A_n,$$

where uniqueness follows from regularity of all factors on the left side of the equation. Analogously B_n is generated by subtraction of the left equation in (3.6) and the right equation in (3.6) multiplied by X_{n-2}^{-1} and Y_{n-2}^{-1}, respectively. Finally the above construction leads to $P_n = X_n$ and $Q_n = Y_n$ for all $n \geq 0$. □

Remark 3.5. From Theorem 3.4 computation of ncf, whose nth denominators are identically E and nth numerators are the nth partial sum of the series $\sum_{k=0}^{\infty} C_k$, the nth partial product of an infinite product is possible without difficulties, under the hypothesis that the respective expressions are invertible elements.

As a consequence of Theorem 3.4 we obtain the following result.

Corollary 3.6. *Let $E + A_1(B_1 + A_2(\ldots)^{-1})^{-1}$ be a ncf with $A_n \in M^*$ for all $n \in \mathbb{N}$, nth numerators $P_n \in M^*$ and denominators $Q_n \in M^*$ for all $n \geq 0$. Then*

$$E - A_1(B_1 + A_1 + A_2(B_2 + A_3(\ldots)^{-1})^{-1})^{-1}$$

has the nth numerators Q_n and denominators P_n.

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