CHAPTER 1

# Interest Rates

DEFINITION 1.1 (Zero-coupon bond). A *zero-coupon* bond with maturity $T > 0$ is a contract that guarantees the holder a cash payment of one unit on the date $T$. The price at time $t \in [0, T]$ of a zero-coupon bond with maturity $T$ is denoted by $P(t, T)$. At time $t$, the *time to maturity* is $T - t$, or, more generally, when taking day-count conventions into account, $\tau(t, T)$.

DEFINITION 1.2 (Interest rates). Let $0 \leq t \leq T < S$.

(i) The *forward rate* for the period $[T, S]$ as seen at time $t$ is defined as

$$R(t; T, S) = -\frac{\ln P(t, S) - \ln P(t, T)}{\tau(T, S)}.$$

(ii) The *continuously-compounded spot interest rate* with maturity $T$ prevailing at $t$ is defined as

$$R(t, T) = -\frac{\ln P(t, T)}{\tau(t, T)}.$$

(iii) The *simply-compounded spot interest rate* with maturity $T$ prevailing at $t$ is defined as

$$L(t, T) = \frac{1 - P(t, T)}{\tau(t, T)P(t, T)}.$$

(iv) The *simply-compounded forward interest rate* for the period $[T, S]$ as seen at time $t$ is defined as

$$F(t; T, S) = \frac{1}{\tau(T, S)} \left( \frac{P(t, T)}{P(t, S)} - 1 \right).$$

(v) The *instantaneous forward interest rate* with maturity $T$ at $t$ is defined as

$$f(t, T) = -\frac{\partial \ln P(t, T)}{\partial T}.$$

(vi) The *instantaneous spot rate* at time $t$ is defined as

$$r(t) = f(t, t).$$

(vii) For $\alpha, \beta \in \mathbb{N}$ with $\alpha < \beta$ and times $T = T_\alpha < \ldots < T_\beta = S$, the *forward swap rate* for $\mathcal{T} = \{T_\alpha, \ldots, T_\beta\}$ at time $t$ is defined as

$$S_{\alpha,\beta}(t) = \frac{P(t,T) - P(t,S)}{\sum_{i=\alpha+1}^{\beta} \tau(T_{i-1}, T_i) P(t, T_i)}.$$

Remark 1.3 (Interest rates). Let $0 \le t \le T < S$.

(i) Suppose $R$ is the equivalent constant rate of interest over the period $[T, S]$. In order to exclude arbitrage, we should have

$$e^{R\tau(T,S)} = \frac{P(t,T)}{P(t,S)}.$$

(ii) We have

$$R(t,T) = R(t;t,T)$$

and

$$P(t,T) = e^{-R(t,T)\tau(t,T)}.$$

Thus $R(t,T)$ is the constant rate at which an investment of $P(t,T)$ units of currency at time $t$ accrues continuously to yield a unit amount of currency at maturity.

(iii) We have

$$P(t,T) = \frac{1}{1 + \tau(t,T)L(t,T)}.$$

Thus $L(t,T)$ is the constant rate at which an investment of $P(t,T)$ units of currency at time $t$ produces one unit amount of currency at maturity $T$, when accruing occurs proportionally to the investment time.

(iv) $F(t;T,S)$ is the value of the fixed rate that makes an FRA for the period between $T$ and $S$ a fair contract at time $t$. FRAs are introduced in Section 2.1. Note also

$$L(t,T) = F(t;t,T).$$

(v) If $\tau(T,S) = S - T$, then we have

$$f(t,T) = \lim_{S \to T^+} F(t;T,S) = \lim_{S \to T^+} R(t;T,S).$$

Note also that for $t \le s \le T$ we have

$$P(t,T) = P(t,s) \exp\left( - \int_s^T f(t,u) \mathrm{d}u \right)$$

and in particular

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\mathrm{d}u\right).$$

(vi) The rate $r(t)$, also briefly called the *short rate*, is the instantaneous rate at which the bank accrues, where the *bank account* is defined as

$$B(t) = \exp\left(\int_0^t r(s)\mathrm{d}s\right).$$

The short rate is also used to define the *discount factor*

$$D(t,T) = \exp\left(-\int_t^T r(s)\mathrm{d}s\right) = \frac{B(t)}{B(T)}.$$

If rates $r(t)$ are deterministic, then so is $D$ and we have necessarily $D(t,T) = P(t,T)$. However, if rates $r(t)$ are stochastic, $D(t,T)$ is stochastic while $P(t,T)$ is deterministic, so they cannot be the same. We will see later that $P(t,T)$ can be viewed as the expectation of the random variable $D(t,T)$ under a particular probability measure. Concerning stochastic rates $r(t)$, we also note that when dealing with interest-rate products, the main variability that matters is clearly that of the interest rates themselves.

(vii) $S_{\alpha,\beta}(t)$ is the value of the fixed rate that makes an IRS for the period between $T$ and $S$ a fair contract at time $t$. IRSs are introduced in Section 2.2. Note also that

$$S_{\beta-1,\beta}(t) = F(t;T_{\beta-1},T_\beta).$$

Finally, put

$$F_j(t) = F(t;T_{j-1},T_j) \quad \text{and} \quad \tau_j = \tau(T_{j-1},T_j)$$

and note that

$$\frac{P(t,T_i)}{P(t,T_\alpha)} = \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}$$

implies

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}.$$