CHAPTER 10

Pricing of Further Derivatives

10.1. In-Arrears Swaps

DEFINITION 10.1 (In-arrears swaps). An *in-arrears swap* of payer type, depending on the *notional value* N, the *fixed rate* K, and the *set of times* $\mathcal{T}$, is a contract, where its holder pays $NK\tau(T_i, T_{i+1})$ and receives $N\tau(T_i, T_{i+1})L(T_i, T_{i+1})$ units of currency at the same time T_i, for all $\alpha + 1 \leq i \leq \beta$.

REMARK 10.2 (In-arrears swaps). An in-arrears swap is an IRS that resets at dates $T_{\alpha+1}, \ldots, T_\beta$ and pays at the same dates, with notional value N and fixed-leg rate K. The discounted payoff of an in-arrears swap of payer type is

$$N \sum_{i=\alpha+1}^{\beta} \left(L(T_i, T_{i+1}) - K\right) \tau_{i+1} D(t, T_i).$$

LEMMA 10.3. *Assume* $0 \leq t \leq T \leq S$. *If* X *is* $\mathcal{F}(T)$*-measurable, then*

$$\mathbb{E}(D(t,T)X|\mathcal{F}(t)) = \mathbb{E}\left(\frac{D(t,S)X}{P(T,S)}\middle| \mathcal{F}(t)\right).$$

THEOREM 10.4 (Price of in-arrears swaps in the LFM model). *In the LFM model, the price of an in-arrears swap of payer type with notional value* N*, fixed rate* K*, and the set of times* $\mathcal{T}$*, is given by*

$$\begin{aligned}\mathrm{IAS}(t, \mathcal{T}, N, K) = N \sum_{i=\alpha+1}^{\beta} \Big\{ P(t, T_{i+1}) \left(1 + 2\tau_{i+1} F_{i+1}(t) + \tau_{i+1}^2 F_{i+1}^2(t) e^{v_{i+1}^2(t)}\right) \\ - (1 + \tau_{i+1} K) P(t, T_i) \Big\},\end{aligned}$$

where

$$v_i(t) = \sqrt{\int_t^{T_{i-1}} \sigma_i^2(u) \mathrm{d}u}, \quad \alpha + 1 \leq i \leq \beta.$$

LEMMA 10.5. *Define*

$$B_i(T,S) = \frac{1 - e^{-k_i(S-T)}}{k_i} \quad \textit{and} \quad B_{ij}(T,S) = \frac{1 - e^{-(k_i + k_j)(S-T)}}{k_i + k_j}$$

for $i, j \in \{1,2\}$. *Then we have*

$$\int_t^{\hat{T}} \left(B_i(u,S) - B_i(u,T)\right)\left(B_j(u,S) - B_j(u,\tilde{T})\right) \mathrm{d}u$$
$$= e^{-k_i(T-\hat{T})} e^{-k_j(\tilde{T}-\hat{T})} B_i(T,S) B_j(\tilde{T},S) B_{ij}(t,\hat{T}).$$

Theorem 10.6 (Price of in-arrears swaps in the G2++ model). *In the G2++ model, the price of an in-arrears swap of payer type with notional value* N*, fixed rate* K*, and the set of times* $\mathcal{T}$*, is given by*

$$\mathrm{IAS}(t,\mathcal{T},N,K) = N \sum_{i=\alpha+1}^{\beta} P(t,T_i) \left(\frac{P(t,T_i) e^{V_{i+1}^2(t)}}{P(t,T_{i+1})} - (1 + K\tau_{i+1}) \right),$$

where

$$\begin{aligned} V_i^2(t) &= \sigma_1^2 B_1^2(T_{i-1},T_i) B_{11}(t,T_{i-1}) + \sigma_2^2 B_2^2(T_{i-1},T_i) B_{22}(t,T_{i-1}) \\ &\quad + 2\sigma_1\sigma_2\rho B_1(T_{i-1},T_i) B_2(T_{i-1},T_i) B_{12}(t,T_{i-1}). \end{aligned}$$

10.2. In-Arrears Caps

Definition 10.7 (In-arrears caps). An *in-arrears cap* of payer type, depending on the *notional value* N, the *cap rate* K, and the *set of times* $\mathcal{T}$, is a contract, where its holder pays $N\tau(T_i,T_{i+1})K$ and receives $N\tau(T_i,T_{i+1})L(T_i,T_{i+1})$ units of currency at the same time T_i, but only if $L(T_i,T_{i+1}) > K$, for all $\alpha+1 \le i \le \beta$.

Remark 10.8 (In-arrears caps). An in-arrears cap is composed by caplets resetting at dates $T_{\alpha+1}, \dots, T_\beta$ and paying at the same dates, with notional value N and cap rate K. The discounted payoff of an in-arrears cap of payer type is

$$N \sum_{i=\alpha+1}^{\beta} \left(L(T_i,T_{i+1}) - K\right)^+ \tau_{i+1} D(t,T_i).$$

Lemma 10.9. *Let* $K > 0$*. If* Y *is lognormally distributed such that* $\mathbb{E}(\ln(Y)) = M$ *and* $\mathbb{V}(\ln(Y)) = V^2$*, then*

$$\mathbb{E}(Y(Y-K)^+) = e^{M+\frac{3V^2}{2}} \,\mathrm{Bl}\left(Ke^{-V^2}, e^{M+\frac{V^2}{2}}, V\right).$$

Theorem 10.10 (Price of in-arrears caps in the LFM model). *In the LFM model, the price of an in-arrears cap of payer type with notional value* N*, cap rate*

K, and the set of times $\mathcal{T}$, is given by

$$\mathrm{IAC}(t,\mathcal{T},N,K) = N \sum_{i=\alpha+1}^{\beta} P(t,T_{i+1})\tau_{i+1}\left\{\mathrm{Bl}\left(K,F_{i+1}(t),v_{i+1}(t)\right)\right.$$
$$\left. +\tau_{i+1}F_{i+1}(t)e^{v_{i+1}^2(t)}\,\mathrm{Bl}\left(Ke^{-v_{i+1}^2(t)},F_{i+1}(t),v_{i+1}(t)\right)\right\},$$

where v_i is as given in Theorem 10.4.

THEOREM 10.11 (Price of in-arrears caps in the G2++ model). *In the G2++ model, the price of an in-arrears cap of payer type with notional value N, fixed rate K, and the set of times $\mathcal{T}$, is given by*

$$\mathrm{IAC}(t,\mathcal{T},N,K) = N \sum_{i=\alpha+1}^{\beta} P(t,T_i)\,\mathrm{Bl}\left(1+K\tau_{i+1},\frac{P(t,T_i)e^{V_{i+1}^2(t)}}{P(t,T_{i+1})},V_{i+1}(t))\right),$$

where $V_i(t)$ is as in Theorem 10.6.

10.3. Caps with Deferred Caplets

DEFINITION 10.12 (Caps with deferred caplets). A *cap with deferred caplets* of payer type, depending on the *notional value* N, the *cap rate* K, and the *set of times* $\mathcal{T}$, is a contract, where its holder pays $N\tau(T_{i-1},T_i)K$ and receives $N\tau(T_{i-1},T_i)L(T_{i-1},T_i)$ units of currency at the same time T_β, but only if $L(T_{i-1},T_i) > K$, for all $\alpha+1 \le i \le \beta$.

REMARK 10.13 (Caps with deferred caplets). Caps with deferred caplets are caps for which all caplets payments occur at the final time T_β. The discounted payoff of a cap with deferred caplets of payer type is

$$N \sum_{i=\alpha+1}^{\beta} \left(L(T_{i-1},T_i)-K\right)^+ \tau_i D(0,T_\beta).$$

DEFINITION 10.14 (Forward-rate dynamics in the partially frozen LFM model). In the *partially frozen LFM model* (with respect to the terminal measure), the simply-compounded forward interest rates F_i is assumed to satisfy the stochastic differential equation

$$\mathrm{d}F_i(t) = \sigma_i(t)F_i(t)\mathrm{d}W^{T_\beta}(t) - \left(\sum_{j=i+1}^{\beta}\frac{\tau_j\sigma_j(t)\sigma_i(t)F_j(0)F_i(t)}{1+\tau_jF_j(0)}\right)\mathrm{d}t,$$

where σ_j are deterministic and W^S is a Brownian motion under the S-forward measure.

Remark 10.15. The dynamics of the forward rate in the partially frozen LFM model is an approximation of the corresponding dynamics in the LFM model, by replacing $F_j(t)$ twice by $F_j(0)$ in the drift of the last formula of Theorem 8.6.

Theorem 10.16 (Price of caps with deferred caplets in the partially frozen LFM model). *In the partially frozen LFM model, the price of a cap with deferred caplets of payer type with notional value N, cap rate K, and the set of times $\mathcal{T}$, is given by*

$$\mathrm{CDC}(0,\mathcal{T},N,K) = N \sum_{i=\alpha+1}^{\beta} P(0,T_i)\tau_i \times$$
$$\times \mathrm{Bl}\left(K, F_i(0)\exp\left(-\sum_{j=i+1}^{\beta} \frac{\tau_j F_j(0)}{1+\tau_j F_j(0)} \int_0^{T_{i-1}} \sigma_i(u)\sigma_j(u)\mathrm{d}u\right), v_i(0)\right),$$

where v_i is as given in Theorem 10.4.

Theorem 10.17 (Price of caps with deferred caplets in the G2++ model). *In the G2++ model, the price of an in-arrears swap of payer type with notional value N, fixed rate K, and the set of times $\mathcal{T}$, is given by*

$$\mathrm{CDC}(t,\mathcal{T},N,K) = N \sum_{i=\alpha+1}^{\beta} P(t,T_\beta)\,\mathrm{Bl}\left(1+\tau_i K, \frac{P(t,T_{i-1})e^{-\tilde{V}_i^2(t)}}{P(t,T_i)}, V_i(t)\right),$$

where $V_i(t)$ is as in Theorem 10.6 and

$$\begin{aligned}\tilde{V}_i^2(t) \quad = \quad & \sigma_1^2 e^{-k_1(T_i-T_{i-1})} B_1(T_{i-1},T_i)B_1(T_i,T_\beta)B_{11}(t,T_{i-1}) \\ & +\sigma_2^2 e^{-k_2(T_i-T_{i-1})} B_2(T_{i-1},T_i)B_2(T_i,T_\beta)B_{22}(t,T_{i-1}) \\ & +\sigma_1\sigma_2\rho\left(B_1(T_{i-1},T_i)B_2(T_i,T_\beta)+B_2(T_{i-1},T_i)B_1(T_i,T_\beta)\right)B_{12}(t,T_{i-1}).\end{aligned}$$