CHAPTER 2

Interest Rate Derivatives

2.1. Forward Rate Agreements

Definition 2.1 (FRA). A *forward rate agreement*, briefly *FRA*, depending on the *notional value* N, the *fixed rate* K, the *expiry time* T, and the *maturity time* $S > T$, is a contract, where its holder receives $N\tau(T,S)K$ and pays $N\tau(T,S)L(T,S)$ units of currency at the same time S.

Remark 2.2 (FRA). An FRA gives its holder an interest-rate payment for the period between T and $S > T$. The contract allows to "lock in" the interest rate between T and S at the desired value K. At maturity S, a fixed payment based on a fixed rate K is exchanged against a floating payment based on the spot rate $L(T,S)$, resetting in T and with maturity S. The value of an FRA at time S is

$$N\tau(T,S)\left(K - L(T,S)\right) = N\left(\tau(T,S)K - \frac{1}{P(T,S)} + 1\right).$$

The value of an FRA at time $t \le T$ is

$$\begin{aligned}\mathrm{FRA}(t,T,S,N,K) &= N\left(\tau(T,S)P(t,S)K - P(t,T) + P(t,S)\right)\\ &= N\tau(T,S)P(t,S)\left(K - F(t;T,S)\right).\end{aligned}$$

Hence $F(t;T,S)$ is that value of K that makes the FRA a fair contract at time t. We also see that in order to value an FRA, we can just replace $L(T,S)$ by $F(t;T,S)$ in the payoff at S and then take the present value at t.

2.2. Interest Rate Swaps

Definition 2.3 (IRS). We consider two kinds of *interest rate swaps*, briefly *IRS*.

(i) A *receiver IRS*, briefly *RFS*, depending on the *notional value* N, the *fixed rate* K, and the *set of times* $\mathcal{T}$, is a contract, where its holder receives

$N\tau(T_{i-1}, T_i)K$ and pays $N\tau(T_{i-1}, T_i)L(T_{i-1}, T_i)$ units of currency at the same time T_i, for all $\alpha + 1 \leq i \leq \beta$.

(ii) A *payer IRS*, briefly *PFS*, depending on the *notional value* N, the *fixed rate* K, and the *set of times* $\mathcal{T}$, is a contract, where its holder pays $NK\tau(T_{i-1}, T_i)$ and receives $N\tau(T_{i-1}, T_i)L(T_{i-1}, T_i)$ units of currency at the same time T_i, for all $\alpha + 1 \leq i \leq \beta$.

Remark 2.4 (IRS). An IRS exchanges interest payments starting from a future time instant $T_{\alpha+1}$. At every instant T_i, the holder of an RFS receives an amount corresponding to a fixed interest rate of the notional value ("fixed leg") and pays an amount corresponding to the LIBOR rate that is reset at the previous time instant T_{i-1} ("floating leg"). The discounted payoff at time $t \leq T_\alpha$ of an RFS is

$$N \sum_{i=\alpha+1}^{\beta} \left(K - L(T_{i-1}, T_i)\right) \tau_i D(t, T_i)$$

and of a PFS is

$$N \sum_{i=\alpha+1}^{\beta} \left(L(T_{i-1}, T_i) - K\right) \tau_i D(t, T_i).$$

The value of an RFS at time $t \leq T_\alpha$ is

$$\begin{aligned} \mathrm{RFS}(t, \mathcal{T}, N, K) &= \sum_{i=\alpha+1}^{\beta} \mathrm{FRA}(t, T_{i-1}, T_i, N, K) \\ &= NP(t, T_\beta) - NP(t, T_\alpha) + NK \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i) \\ &= N\left(K - S_{\alpha,\beta}(t)\right) \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i). \end{aligned}$$

Hence $S_{\alpha,\beta}(t)$ is that value of K that makes the IRS a fair contract at time t. We also see that an IRS can be viewed as a portfolio of a coupon-bearing bond (fixed leg) and a floating-rate note (floating leg).

Definition 2.5 (Floating-rate note). A *floating-rate note*, briefly *FRN*, depending on the set of *future times* $\mathcal{T}$ and the *notional value* N, is a contract, where its holder receives $N\tau(T_{i-1}, T_i)L(T_{i-1}, T_i)$ units of currency at time T_i for all $\alpha + 1 \leq i \leq \beta$. In addition, the holder also receives N units of currency at time T_β.

REMARK 2.6 (Floating-rate note). A floating-rate note ensures payments at future times $T_{\alpha+1}, \ldots, T_\beta$ of the LIBOR rates that reset at the previous instants $T_\alpha, \ldots, T_{\beta-1}$ and, moreover, it pays a last cash flow consisting of the reimbursement of the notional value of the note at the final time. The value of a floating-rate note at time $t \leq T_\alpha$ is

$$\mathrm{FRN}(t, \mathcal{T}, N) = NP(t, T_\alpha).$$

This means that a floating-rate note with notional value N at its first reset date is always worth its notional value, i.e., “a floating-rate note trades at par”.

DEFINITION 2.7 (Coupon-bearing bond). A *coupon-bearing bond*, depending on the set of *future times* $\mathcal{T}$ and the deterministic *cash flow* $c = \{c_{\alpha+1}, \ldots, c_\beta\}$, is a contract, where its holder receives c_i units of currency at time T_i for all $\alpha + 1 \leq i \leq \beta$.

REMARK 2.8 (Coupon-bearing bond). The value of a coupon-bearing bond at time $t \leq T_\alpha$ is

$$\mathrm{CB}(t, \mathcal{T}, c) = \sum_{i=\alpha+1}^{\beta} c_i P(t, T_i).$$

If we let

$$c_i = N\tau_i K \quad \text{for} \quad \alpha + 1 \leq i \leq \beta - 1 \quad \text{and} \quad c_\beta = N\tau_\beta K + N,$$

the value is

$$\mathrm{CB}(t, \mathcal{T}, c) = NP(t, T_\beta) + NK \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i),$$

and then

$$\mathrm{RFS}(t, \mathcal{T}, N, K) = \mathrm{CB}(t, \mathcal{T}, c) - \mathrm{FRN}(t, \mathcal{T}, N).$$

2.3. Interest Rate Caps and Floors

DEFINITION 2.9 (Caplets and floorlets). (i) A *floorlet*, depending on the *notional value* N, the *floor rate* K, the *expiry time* T, and the *maturity time* $S > T$, is a contract, where its holder receives $N\tau(T, S)K$ and pays $N\tau(T, S)L(T, S)$ units of currency at the same time S, but only if $L(T, S) < K$.

(ii) A *caplet*, depending on the *notional value* N, the *cap rate* K, the *expiry time* T, and the *maturity time* $S > T$, is a contract, where its holder pays $NK\tau(T,S)$ and receives $N\tau(T,S)L(T,S)$ units of currency at the same time S, but only if $L(T,S) > K$.

REMARK 2.10 (Caplets and floorlets). A floorlet gives its holder an interest-rate payment for the period between T and $S > T$. At maturity S, a fixed payment based on a fixed rate K is exchanged against a floating payment based on the spot rate $L(T,S)$, resetting in T and with maturity S. However, this is done only if the spot rate does not exceed K. Hence the holder of a floorlet receives interest at a rate which is at least K. The discounted payoff at time $t \le T$ of a floorlet is

$$N\left(K - L(T,S)\right)^{+} \tau(T,S)D(t,S).$$

Similarly, the discounted payoff at time $t \le T$ of a caplet is

$$N\left(L(T,S) - K\right)^{+} \tau(T,S)D(t,S),$$

meaning that the holder of a caplet is paying interest at a rate which is at most K, i.e., the interest rate is *capped* to the fixed cap rate K. Hence caplets and floorlets are (call and put) options on interest rates, and they can be priced with Black's formula. This will be done in Section 8.1.

DEFINITION 2.11 (Caps and floors).

(i) A *floor*, depending on the *notional value* N, the *floor rate* K, and the *set of times* $\mathcal{T}$, is a contract, where its holder receives $N\tau(T_{i-1},T_i)K$ and pays $N\tau(T_{i-1},T_i)L(T_{i-1},T_i)$ units of currency at the same time T_i, but only if $L(T_{i-1},T_i) < K$, for all $\alpha + 1 \le i \le \beta$.

(ii) A *cap*, depending on the *notional value* N, the *cap rate* K, and the *set of times* $\mathcal{T}$, is a contract, where its holder pays $N\tau(T_{i-1},T_i)K$ and receives $N\tau(T_{i-1},T_i)L(T_{i-1},T_i)$ units of currency at the same time T_i, but only if $L(T_{i-1},T_i) > K$, for all $\alpha + 1 \le i \le \beta$.

REMARK 2.12 (Caps and floors). A floor is an RFS where each exchange payment is executed only if it has positive value. It can also be considered as a portfolio of floorlets. The discounted payoff at time $t \leq T_\alpha$ of a floor is

$$N \sum_{i=\alpha+1}^{\beta} (K - L(T_{i-1}, T_i))^+ \tau_i D(t, T_i).$$

Similarly, a cap is a PFS where each exchange payment is executed only if it has positive value. It can also be considered as a portfolio of caplets. The discounted payoff at time $t \leq T_\alpha$ of a cap is

$$N \sum_{i=\alpha+1}^{\beta} (L(T_{i-1}, T_i) - K)^+ \tau_i D(t, T_i).$$

Caps and floors can be priced with a sum of Black's formulas. This will be done in Section 8.1.

Definition 2.13 (ATM). Let $K_{\text{ATM}} = S_{\alpha,\beta}(t)$. A cap or floor is said to be *at the money*, briefly *ATM* if $K = K_{\text{ATM}}$. A cap is called *in the money*, briefly *ITM* if $K < K_{\text{ATM}}$, while a floor is said to be ITM if $K > K_{\text{ATM}}$. A cap is called *out of the money*, briefly *OTM* if $K > K_{\text{ATM}}$, while a floor is said to be ITM if $K < K_{\text{ATM}}$.

2.4. Swaptions

Definition 2.14 (Swaptions). A *swap option*, briefly *swaption*, is an option on an IRS. The time T_α is called the *swaption maturity*. The underlying IRS length $T_\beta - T_\alpha$ is called the *tenor* of the swaption.

(i) A European *payer swaption* is a contract that gives the holder the right (but no obligation) to enter a PFS at the swaption maturity.

(ii) A European *receiver swaption* is a contract that gives the holder the right (but no obligation) to enter an RFS at the swaption maturity.

Remark 2.15 (Swaption). The value of the underlying IRS of a payer swaption at time T_α is

$$N \sum_{i=\alpha+1}^{\beta} (F(T_\alpha; T_{i-1}, T_i) - K)\tau_i P(T_\alpha, T_i).$$

The discounted payer-swaption payoff therefore is

$$N \left(\sum_{i=\alpha+1}^{\beta} (F(T_\alpha; T_{i-1}, T_i) - K)\tau_i P(T_\alpha, T_i) \right)^+ D(t, T_\alpha).$$

Thus it is not possible to decompose the payer-swaption payoff as can be done for caps. However, the value of a payer swaption is smaller than or equal to the value of the corresponding cap contract, due to the inequality

$$\left(\sum_{i=\alpha+1}^{\beta} (F(T_\alpha; T_{i-1}, T_i) - K)\tau_i P(T_\alpha, T_i) \right)^+ \\ \leq \sum_{i=\alpha+1}^{\beta} (F(T_\alpha; T_{i-1}, T_i) - K)^+ \tau_i P(T_\alpha, T_i).$$

Swaptions can be priced with a Black-like formula. This will be done in Section 8.2.

Definition 2.16 (ATM). Both payer and receiver swaption are said to be *ATM* if $K = K_{\text{ATM}}$. A payer swaption is called *ITM* if $K < K_{\text{ATM}}$, and a receiver swaption is said to be ITM if $K > K_{\text{ATM}}$. A payer swaption is called *OTM* if $K > K_{\text{ATM}}$, and a receiver swaption is said to be OTM if $K < K_{\text{ATM}}$.