CHAPTER 3

Pricing Theory

3.1. Preliminaries from Financial Mathematics

In this section we give some crucial preliminaries from financial mathematics. The results are formulated in a nonrigorous way without stating the precise assumptions.

We start by giving the solutions to two important stochastic differential equations.

Theorem 3.1 (Linear stochastic differential equation with deterministic diffusion coefficient). *The solution of the linear stochastic differential equation*

$$\mathrm{d}X(t) = (\alpha(t) + \beta(t)X(t))\mathrm{d}t + \gamma(t)\mathrm{d}W(t),$$

where α, β, and γ are deterministic functions, is given by

$$X(t) = X(s)e^{\int_s^t \beta(\tau)\mathrm{d}\tau} + \int_s^t \alpha(u)e^{\int_u^t \beta(\tau)\mathrm{d}\tau}\mathrm{d}u + \int_s^t \gamma(u)e^{\int_u^t \beta(\tau)\mathrm{d}\tau}\mathrm{d}W(u).$$

Theorem 3.2 (Geometric Brownian motion). *The solution of the lognormal linear stochastic differential equation*

$$\mathrm{d}X(t) = \mu(t)X(t)\mathrm{d}t + \sigma(t)X(t)\mathrm{d}W(t),$$

where μ and σ are deterministic functions, is given by the generalized geometric Brownian motion

$$X(t) = X(s)\exp\left\{\int_s^t \left(\mu(\tau) - \frac{\sigma^2(\tau)}{2}\right)\mathrm{d}\tau + \int_s^t \sigma(\tau)\mathrm{d}W(\tau)\right\}.$$

In order to check solutions of stochastic differential equations, Itô's formula and the stochastic Leibniz rule as given next are useful.

Theorem 3.3 (Itô's formula).

$$\mathrm{d}F(t, X(t)) = F_t(t, X(t))\mathrm{d}t + F_x(t, X(t))\mathrm{d}X(t) + \frac{1}{2}F_{xx}(t, X(t))(\mathrm{d}X(t))^2,$$

where $(\mathrm{d}X(t))^2$ *can be calculated by formally squaring* $\mathrm{d}X(t)$ *and using the "identities"*

$$(\mathrm{d}t)^2 = (\mathrm{d}t)(\mathrm{d}W(t)) = 0 \quad \textit{and} \quad (\mathrm{d}W(t))^2 = \mathrm{d}t.$$

THEOREM 3.4 (Stochastic Leibniz rule).

$$\mathrm{d}(X(t)Y(t)) = X(t)(\mathrm{d}Y(t)) + (\mathrm{d}X(t))Y(t) + (\mathrm{d}X(t))(\mathrm{d}Y(t)).$$

The next theorem helps to determine distribution, expectation, and variance of solutions to certain stochastic differential equations.

THEOREM 3.5 (Itô integral of deterministic integrand). *If* f *is a deterministic function, then for* $0 \le s \le t$*,*

$$I(t) = \int_s^t f(u)\mathrm{d}W(u),$$

conditionally on $\mathcal{F}(s)$*, is normally distributed with*

$$\mathbb{E}(I(t)|\mathcal{F}(s)) = 0 \quad \textit{and} \quad \mathbb{V}(I(t)|\mathcal{F}(s)) = \int_s^t f^2(u)\mathrm{d}u.$$

Now we give the theorem about the change of measure.

THEOREM 3.6 (Girsanov). *Consider the stochastic differential equation*

$$\mathrm{d}X(t) = \mu(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t),$$

where W *is a Brownian motion under the probability measure* $\mathbb{P}$*. Let be given a new drift* μ^**, define*

$$\lambda(t) = \frac{\mu^*(X(t)) - \mu(X(t))}{\sigma(X(t))},$$

and define the probability measure $\mathbb{P}^*$ *by*

$$\left.\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}(t)} = \exp\left\{-\frac{1}{2}\int_0^t \lambda^2(s)\mathrm{d}s + \int_0^t \lambda(s)\mathrm{d}W(s)\right\}.$$

Then $\mathbb{P}^* \sim \mathbb{P}$ *and the process* W^* *defined by*

$$\mathrm{d}W^*(t) = \mathrm{d}W(t) - \lambda(t)\mathrm{d}t$$

is a Brownian motion under $\mathbb{P}^*$ *and we have*

$$\mathrm{d}X(t) = \mu^*(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W^*(t).$$

The following theorem summarizes the change of numéraire technique.

THEOREM 3.7 (Change of numéraire). *If $\mathbb{Q} = \mathbb{Q}^B$ is a risk-neutral measure, i.e., X/B is a martingale under $\mathbb{Q}^B$ for any traded asset X, then for each numéraire N, i.e., any positive nondividend-paying asset, there exists a measure $\mathbb{Q}^N \sim \mathbb{Q}^B$, defined by*

$$\left.\frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{Q}^B}\right|_{\mathcal{F}(t)} = \frac{X(t)}{X(0)B(t)},$$

under which X/N is a martingale for any traded asset X, i.e.,

$$\mathbb{E}^N\left(\frac{X(T)}{N(T)}\middle|\mathcal{F}(t)\right) = \frac{X(t)}{N(t)} \quad \text{for all} \quad 0 \le t \le T,$$

i.e., X/N has no drift under $\mathbb{Q}^N$.

We finally also require the following stochastic representation of solutions of partial differential equations.

THEOREM 3.8 (Feynmann–Kac). *The solution of the partial differential equation*

$$g_t + \mu g_x + \frac{\sigma^2}{2} g_{xx} = 0$$

with final condition $g(T,x) = h(x)$ has the stochastic representation

$$g(t,x) = \mathbb{E}(h(X(T))|X(t) = x),$$

where X satisfies the stochastic differential equation

$$\mathrm{d}X(s) = \mu(X(s))\mathrm{d}s + \sigma(X(s))\mathrm{d}W(s)$$

with initial condition $X(t) = x$.

3.2. Three Examples of Numéraires

EXAMPLE 3.9 (The bank account as numéraire). Any traded asset X has drift $r(t)X(t)$ under the risk-neutral measure $\mathbb{Q} = \mathbb{Q}^B$.

EXAMPLE 3.10 (The zero-coupon bond as numéraire). Let $\mathbb{Q}^S$ be the *S-forward measure*, i.e., the measure associated to the numéraire $P(t,S)$. Then $F(t;T,S)$ is a martingale under $\mathbb{Q}^S$.

EXAMPLE 3.11 (A portfolio of zero-coupon bonds as numéraire). Let $\mathbb{Q}^{\alpha,\beta}$ be the measure associated to the numéraire $\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$. Then $S_{\alpha,\beta}(t)$ is a martingale under $\mathbb{Q}^{\alpha,\beta}$.

3.3. Pricing Formulas

THEOREM 3.12 (Risk-neutral pricing). *For any traded asset X, we have*

$$X(t) = \mathbb{E}(X(T)D(t,T)|\mathcal{F}(t)).$$

THEOREM 3.13 (Zero-coupon bond price). *We have*

$$P(t,T) = \mathbb{E}(D(t,T)|\mathcal{F}(t)).$$

REMARK 3.14 (Zero-coupon bond price). The price of a zero-coupon bond with maturity T at time $t \in [0,T]$ is equal to the conditional expectation of the discount factor from T to t with respect to $\mathcal{F}(t)$ under the risk-neutral measure.

THEOREM 3.15 (T-forward measure pricing). *For any traded asset X, we have*

$$X(t) = P(t,T)\mathbb{E}^T(X(T)|\mathcal{F}(t)).$$

THEOREM 3.16 (Martingale property of the forward rate). *We have*

$$\mathbb{E}^S(F(t;T,S)|\mathcal{F}(u)) = F(u;T,S) \quad \textit{for all} \quad 0 \le u \le t \le T \le S.$$

REMARK 3.17 (Martingale property of the forward rate). Any simply-compounded forward interest rate spanning a time interval ending in S is a martingale under the S-forward measure.

COROLLARY 3.18. *We have*

$$\mathbb{E}^S(L(T,S)|\mathcal{F}(t)) = F(t;T,S) \quad \textit{for all} \quad 0 \le t \le T \le S.$$

THEOREM 3.19 (Expectation of short rate under the forward measure). *We have*

$$\mathbb{E}^T(r(T)|\mathcal{F}(t)) = f(t,T) \quad \textit{for all} \quad 0 \le t \le T.$$

REMARK 3.20 (Expectation of short rate under the forward measure). The expectation of any future instantaneous spot rate under the corresponding forward measure is equal to the related instantaneous forward interest rate.

THEOREM 3.21 (Pricing of European options on zero-coupon bonds). *The price of a European call option with maturity T, strike K, and written on a zero-coupon* *bond with maturity $S > T$ is*

$$\begin{aligned}\mathrm{ZBC}(t,T,S,K) &= \mathbb{E}\left(e^{-\int_t^T r(u)\mathrm{d}u}(P(T,S)-K)^+|\mathcal{F}(t)\right)\\ &= P(t,T)\mathbb{E}^T((P(T,S)-K)^+|\mathcal{F}(t))\end{aligned}$$

for a call and

$$\begin{aligned}\mathrm{ZBP}(t,T,S,K) &= \mathbb{E}\left(e^{-\int_t^T r(u)\mathrm{d}u}(K-P(T,S))^+|\mathcal{F}(t)\right)\\ &= P(t,T)\mathbb{E}^T((K-P(T,S))^+|\mathcal{F}(t))\end{aligned}$$

for a put.

THEOREM 3.22 (Pricing of caplets and floorlets). *The price of a caplet with notional value N, cap rate K, expiry time T, and maturity time $S > T$, is given by*

$$\mathrm{Cpl}(t,T,S,N,K) = N'\,\mathrm{ZBP}(t,T,S,K'),$$

while the price of a floorlet with notional value N, floor rate K, expiry time T, and maturity time $S > T$, is given by

$$\mathrm{Fll}(t,T,S,N,K) = N'\,\mathrm{ZBC}(t,T,S,K'),$$

where

$$N' = N(1+\tau(T,S)K) \quad \text{and} \quad K' = \frac{1}{1+\tau(T,S)K}.$$

THEOREM 3.23 (Pricing of caps and floors). *The price of a cap with notional value N, cap rate K, and the set of times $\mathcal{T}$, is given by*

$$\mathrm{Cap}(t,\mathcal{T},N,K) = \sum_{i=\alpha+1}^{\beta} N_i'\,\mathrm{ZBP}(t,T_{i-1},T_i,K_i'),$$

while the price of a floor with notional value N, floor rate K, and the set of times $\mathcal{T}$, is given by

$$\mathrm{Flr}(t,\mathcal{T},N,K) = \sum_{i=\alpha+1}^{\beta} N_i'\,\mathrm{ZBC}(t,T_{i-1},T_i,K_i'),$$

where

$$N_i' = N(1+\tau_i K) \quad \text{and} \quad K_i' = \frac{1}{1+\tau_i K} \quad \text{for} \quad \alpha+1 \le i \le \beta.$$

3.4. Two Useful Formulas

THEOREM 3.24. *If Y is normally distributed with $\mathbb{E}(Y) = \mu$ and $\mathbb{V}(Y) = \sigma^2$, then*

$$\mathbb{E}(e^Y) = e^{\mu + \frac{\sigma^2}{2}}.$$

THEOREM 3.25. *Let $K > 0$. If Y is lognormally distributed such that $\mathbb{E}(\ln(Y)) = M$ and $\mathbb{V}(\ln(Y)) = V^2$, then*

$$\mathbb{E}((Y-K)^+) = e^{M + \frac{V^2}{2}} \Phi\left(\frac{M - \ln(K) + V^2}{V}\right) - K\Phi\left(\frac{M - \ln(K)}{V}\right),$$

where Φ is the cdf of the standard normal distribution, i.e.,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \mathrm{d}t.$$