CHAPTER 4

One-Factor Short-Rate Models

4.1. Vasicek Model

Definition 4.1 (Short-rate dynamics in the Vasicek model). In the *Vasicek model*, the short rate is assumed to satisfy the stochastic differential equation

$$\mathrm{d}r(t) = k(\theta - r(t))\mathrm{d}t + \sigma \mathrm{d}W(t),$$

where $k, \theta, \sigma > 0$ and W is a Brownian motion under the risk-neutral measure.

Theorem 4.2 (Short rate in the Vasicek model). *Let $0 \leq s \leq t \leq T$. The short rate in the Vasicek model is given by*

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{-k(t-u)}\mathrm{d}W(u)$$

and is, conditionally on $\mathcal{F}(s)$, normally distributed with

$$\mathbb{E}(r(t)|\mathcal{F}(s)) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

and

$$\mathbb{V}(r(t)|\mathcal{F}(s)) = \frac{\sigma^2}{2k}\left(1 - e^{-2k(t-s)}\right).$$

Remark 4.3 (Short rate in the Vasicek model). The short rate $r(t)$, for each time t, can be negative with positive probability. This is a major drawback of the Vasicek model. On the other hand, the short rate in the Vasicek model is *mean reverting*, i.e., rates revert to a long-time level, since

$$\mathbb{E}(r(t)) \to \theta \quad \text{as} \quad t \to \infty.$$

Theorem 4.4 (Zero-coupon bond in the Vasicek model). *In the Vasicek model, the price of a zero-coupon bond with maturity T at time $t \in [0, T]$ is given by*

$$P(t, T) = A(t, T)e^{-r(t)B(t,T)},$$

where

$$B(t,T) = \frac{1 - e^{-k(T-t)}}{k}$$

and

$$A(t,T) = \exp\left\{ \left(\theta - \frac{\sigma^2}{2k^2} \right) (B(t,T) - T + t) - \frac{\sigma^2}{4k} B^2(t,T) \right\}.$$

THEOREM 4.5 (Bond-price dynamics in the Vasicek model). *In the Vasicek model, the price of a zero-coupon bond with maturity T satisfies the stochastic differential equations*

$$\mathrm{d}P(t,T) = r(t)P(t,T)\mathrm{d}t - \sigma B(t,T)P(t,T)\mathrm{d}W(t)$$

and

$$\mathrm{d}\frac{1}{P(t,T)} = \frac{\sigma^2 B^2(t,T) - r(t)}{P(t,T)}\mathrm{d}t + \frac{\sigma B(t,T)}{P(t,T)}\mathrm{d}W(t).$$

THEOREM 4.6 (T-forward measure dynamics of the short rate in the Vasicek model). *Under the T-forward measure $\mathbb{Q}^T$, the short rate r in the Vasicek model satisfies*

$$\mathrm{d}r(t) = \left[k\theta - \sigma^2 B(t,T) - kr(t)\right]\mathrm{d}t + \sigma \mathrm{d}W^T(t),$$

where the $\mathbb{Q}^T$-Brownian motion W^T is defined by

$$\mathrm{d}W^T(t) = \mathrm{d}W(t) + \sigma B(t,T)\mathrm{d}t.$$

Let $0 \leq s \leq t \leq T$. Then r is given by

$$r(t) = r(s)e^{-k(t-s)} + M^T(s,t) + \sigma \int_s^t e^{-k(t-u)}\mathrm{d}W(u),$$

where

$$M^T(s,t) = \left(\theta - \frac{\sigma^2}{k^2}\right)\left(1 - e^{-k(t-s)}\right) + \frac{\sigma^2}{2k^2}\left(e^{-k(T-t)} - e^{-k(T+t-2s)}\right),$$

and is, conditionally on $\mathcal{F}(s)$, normally distributed with

$$\mathbb{E}^T(r(t)|\mathcal{F}(s)) = r(s)e^{-k(t-s)} + M^T(s,t)$$

and

$$\mathbb{V}^T(r(t)|\mathcal{F}(s)) = \frac{\sigma^2}{2k}\left(1 - e^{-2k(t-s)}\right).$$

THEOREM 4.7 (Forward-rate dynamics in the Vasicek model). *In the Vasicek model, the instantaneous forward interest rate with maturity T is given by*

$$f(t,T) = \left(\theta k - \frac{\sigma^2}{2}B(t,T)\right)B(t,T) + r(t)e^{-k(T-t)}$$

and satisfies the stochastic differential equation

$$\mathrm{d}f(t,T) = \sigma e^{-k(T-t)}\mathrm{d}W^T(t).$$

THEOREM 4.8 (Forward-rate dynamics in the Vasicek model). *In the Vasicek model, the simply-compounded forward interest rate for the period $[T,S]$ satisfies the stochastic differential equation*

$$\mathrm{d}F(t;T,S) = \sigma\left(F(t;T,S) + \frac{1}{\tau(T,S)}\right)(B(t,S) - B(t,T))\mathrm{d}W^S(t).$$

THEOREM 4.9 (Option on a zero-coupon bond in the Vasicek model). *In the Vasicek model, the price of a European call option with strike K and maturity T and written on a zero-coupon bond with maturity S at time $t \in [0,T]$ is given by*

$$\mathrm{ZBC}(t,T,S,K) = P(t,S)\Phi(h) - KP(t,T)\Phi(h-\tilde{\sigma}),$$

where

$$\tilde{\sigma} = \sigma\sqrt{\frac{1-e^{-2k(T-t)}}{2k}}B(T,S)$$

and

$$h = \frac{1}{\tilde{\sigma}}\ln\left(\frac{P(t,S)}{P(t,T)K}\right) + \frac{\tilde{\sigma}}{2}.$$

The price of a corresponding put option is given by

$$\mathrm{ZBP}(t,T,S,K) = KP(t,T)\Phi(-h+\tilde{\sigma}) - P(t,S)\Phi(-h).$$

THEOREM 4.10 (Caps and floors in the Vasicek model). *In the Vasicek model, the price of a cap with notional value N, cap rate K, and the set of times $\mathcal{T}$, is given by*

$$\mathrm{Cap}(t,\mathcal{T},N,K) = N\sum_{i=\alpha+1}^{\beta}\left[P(t,T_{i-1})\Phi(-h_i+\tilde{\sigma}_i) - (1+\tau_i K)P(t,T_i)\Phi(-h_i)\right],$$

while the price of a floor with notional value N, floor rate K, and the set of times $\mathcal{T}$, is given by

$$\mathrm{Flr}(t,\mathcal{T},N,K) = N\sum_{i=\alpha+1}^{\beta}\left[(1+\tau_i K)P(t,T_i)\Phi(h_i) - P(t,T_{i-1})\Phi(h_i-\tilde{\sigma}_i)\right],$$

where

$$\tilde{\sigma}_i = \sigma\sqrt{\frac{1-e^{-2k(T_{i-1}-t)}}{2k}}B(T_{i-1},T_i)$$

and

$$h_i = \frac{1}{\tilde{\sigma}_i}\ln\left(\frac{(1+\tau_i K)P(t,T_i)}{P(t,T_{i-1})}\right) + \frac{\tilde{\sigma}_i}{2}.$$

4.2. Exponential Vasicek Model

Definition 4.11 (Exponential Vasicek model). In the *exponential Vasicek model*, the short rate is given by

$$r(t) = e^{y(t)} \quad \text{with} \quad \mathrm{d}y(t) = k(\theta - y(t))\mathrm{d}t + \sigma \mathrm{d}W(t),$$

where $k, \theta, \sigma > 0$ and W is a Brownian motion under the risk-neutral measure.

Theorem 4.12 (Short rate in the exponential Vasicek model). *The short rate in the exponential Vasicek model satisfies the stochastic differential equation*

$$\mathrm{d}r(t) = \left(k\theta + \frac{\sigma^2}{2} - k\ln(r(t))\right)r(t)\mathrm{d}t + \sigma r(t)\mathrm{d}W(t).$$

Let $0 \le s \le t \le T$. Then r is given by

$$r(t) = \exp\left\{\ln(r(s))e^{-k(t-s)} + \theta\left(1-e^{-k(t-s)}\right) + \sigma\int_s^t e^{-k(t-u)}\mathrm{d}W(u)\right\}$$

and is, conditionally on $\mathcal{F}(s)$, lognormally distributed with

$$\mathbb{E}(r(t)|\mathcal{F}(s)) = \exp\left\{\ln(r(s))e^{-k(t-s)} + \theta\left(1-e^{-k(t-s)}\right) + \frac{\sigma^2}{4k}\left(1-e^{-2k(t-s)}\right)\right\}$$

and

$$\begin{aligned}\mathbb{V}(r(t)|\mathcal{F}(s)) = \exp\left\{2\ln(r(s))e^{-k(t-s)} + 2\theta\left(1-e^{-k(t-s)}\right)\right\} \times \\ \times \exp\left\{\frac{\sigma^2}{2k}\left(1-e^{-2k(t-s)}\right)\right\}\left[\exp\left\{\frac{\sigma^2}{2k}\left(1-e^{-2k(t-s)}\right)\right\}-1\right].\end{aligned}$$

Remark 4.13 (Short rate in the exponential Vasicek model). Since the short rate r in the exponential Vasicek model is lognormally distributed, it is always positive. A disadvantage is that $P(t,T)$ cannot be calculated explicitly. An advantage of the exponential Vasicek model is that r is always mean reverting with

$$\mathbb{E}(r(t)|\mathcal{F}(s)) \to \exp\left(\theta + \frac{\sigma^2}{4k}\right) \quad \text{as} \quad t \to \infty$$

and

$$\mathbb{V}(r(t)|\mathcal{F}(s)) \to \exp\left(2\theta + \frac{\sigma^2}{2k}\right)\left[\exp\left(\frac{\sigma^2}{2k}\right)-1\right] \quad \text{as} \quad t \to \infty.$$

4.3. Dothan Model

Definition 4.14 (Dothan model). In the *Dothan model*, the short rate is assumed to satisfy the stochastic differential equation

$$\mathrm{d}r(t) = ar(t)\mathrm{d}t + \sigma r(t)\mathrm{d}W(t),$$

where $\sigma > 0$, $a \in \mathbb{R}$, and W is a Brownian motion under the risk-neutral measure.

Theorem 4.15 (Short rate in the Dothan model). *Let $0 \le s \le t \le T$. The short rate in the Dothan model is given by*

$$r(t) = r(s)\exp\left\{\left(a - \frac{\sigma^2}{2}\right)(t-s) + \sigma(W(t) - W(s))\right\}$$

and is, conditionally on $\mathcal{F}(s)$, lognormally distributed with

$$\mathbb{E}(r(t)|\mathcal{F}(s)) = r(s)e^{a(t-s)}$$

and

$$\mathbb{V}(r(t)|\mathcal{F}(s)) = r^2(s)e^{2a(t-s)}\left(e^{\sigma^2(t-s)} - 1\right).$$

Remark 4.16 (Short rate in the Dothan model). Since the short rate r in the Dothan model is lognormally distributed, it is always positive. Another advantage is that $P(t,T)$ can be calculated explicitly, although the formula is not as nice and involves the hyperbolic sine, the gamma function, and the modified Bessel function of the second kind. A disadvantage of the Dothan model is that r is mean reverting only iff $a < 0$, and then the mean reversion level is zero.

4.4. Cox–Ingersoll–Ross Model

Definition 4.17 (CIR model). In the *Cox–Ingersoll–Ross model*, briefly *CIR model*, the short rate is assumed to satisfy the stochastic differential equation

$$\mathrm{d}r(t) = k(\theta - r(t))\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W(t),$$

where $k, \theta, \sigma > 0$ with $2k\theta > \sigma^2$ and W is a Brownian motion under the risk-neutral measure.

THEOREM 4.18 (Short rate in the CIR model). *Let $0 \leq s \leq t \leq T$. The short rate in the CIR model satisfies*

$$\mathbb{E}(r(t)|\mathcal{F}(s)) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

and

$$\mathbb{V}(r(t)|\mathcal{F}(s)) = \frac{\sigma^2 r(s)}{k}\left(e^{-k(t-s)} - e^{-2k(t-s)}\right) + \frac{\sigma^2\theta}{2k}\left(1 - e^{-k(t-s)}\right)^2.$$

REMARK 4.19 (Short rate in the CIR model). The prcess followed by the short rate in the CIR model is also sometimes called a *square-root process*. The nice mean reversion property in the Vasicek model is preserved in the CIR model. The bad property of possible negativity in the Vasicek model is removed in the CIR model by assuming $2k\theta > \sigma^2$ and hence ensuring that the origin is inaccessible to the process. On the other hand, the distribution of the short rate in the CIR model is neither normal nor lognormal but it possesses a noncentral chi-squared distribution.

THEOREM 4.20 (Zero-coupon bond in the CIR model). *In the CIR model, the price of a zero-coupon bond with maturity T at time $t \in [0, T]$ is given by*

$$P(t,T) = A(t,T)e^{-r(t)B(t,T)},$$

where

$$A(t,T) = \left(\frac{2he^{(h+k)(T-t)/2}}{2h + (h+k)(e^{h(T-t)} - 1)}\right)^{2k\theta/\sigma^2}$$

and

$$B(t,T) = \frac{2(e^{h(T-t)} - 1)}{2h + (h+k)(e^{h(T-t)} - 1)}$$

with

$$h = \sqrt{k^2 + 2\sigma^2}.$$

THEOREM 4.21 (Bond-price dynamics in the CIR model). *In the CIR model, the price of a zero-coupon bond with maturity T satisfies the stochastic differential equations*

$$\mathrm{d}P(t,T) = r(t)P(t,T)\mathrm{d}t - \sigma\sqrt{r(t)}B(t,T)P(t,T)\mathrm{d}W(t)$$

and

$$\mathrm{d}\frac{1}{P(t,T)} = \frac{(\sigma^2 B^2(t,T) - 1)r(t)}{P(t,T)}\mathrm{d}t + \frac{\sigma\sqrt{r(t)}B(t,T)}{P(t,T)}\mathrm{d}W(t).$$

THEOREM 4.22 (T-forward measure dynamics of the short rate in the CIR model). *Under the T-forward measure $\mathbb{Q}^T$, the short rate r in the CIR model satisfies*

$$\mathrm{d}r(t) = \left[k\theta - (k + \sigma^2 B(t,T))r(t)\right] \mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W^T(t),$$

where the $\mathbb{Q}^T$-Brownian motion W^T is defined by

$$\mathrm{d}W^T(t) = \mathrm{d}W(t) + \sigma B(t,T)\sqrt{r(t)}\mathrm{d}t.$$

THEOREM 4.23 (Forward-rate dynamics in the CIR model). *In the CIR model, the instantaneous forward interest rate with maturity T is given by*

$$f(t,T) = k\theta B(t,T) + r(t)B_T(t,T)$$

and satisfies the stochastic differential equation

$$\mathrm{d}f(t,T) = \sigma\sqrt{r(t)}B_T(t,T)\mathrm{d}W^T(t).$$

THEOREM 4.24 (Forward-rate dynamics in the CIR model). *Let $0 \le t \le T \le S$. In the CIR model, the forward rate $F(t;T,S)$ satisfies*

$$\mathrm{d}F(t;T,S) = \sigma\left(F(t;T,S) + \frac{1}{\tau(T,S)}\right) \times$$
$$\times\sqrt{(B(t,S) - B(t,T))\ln\left((\tau(T,S)F(t;T,S) + 1)\,\frac{A(t,S)}{A(t,T)}\right)}\mathrm{d}W^S(t).$$

4.5. Affine Term-Structure Models

DEFINITION 4.25 (Affine term structure). If bond prices are given by

$$P(t,T) = A(t,T)e^{-r(t)B(t,T)} \quad \text{for all} \quad 0 \le t \le T,$$

where A and B are deterministic functions, then we say that the model possesses an *affine term structure.*

THEOREM 4.26 (Zero-coupon price in affine term-structure models). *In an affine term-structure model in which the short rate satisfies the stochastic differential equation*

$$\mathrm{d}r(t) = (\alpha(t) - \beta(t)r(t))\mathrm{d}t + \sqrt{\gamma(t) + \delta(t)r(t)}\mathrm{d}W(t),$$

we have that P is given by

$$P(t,T) = A(t,T)e^{-r(t)B(t,T)},$$

where A and B are solutions of the system of ordinary differential equations

$$B_t(t,T) = \beta(t)B(t,T) + \frac{\delta(t)}{2}B^2(t,T) - 1, \quad B(T,T) = 0,$$
$$A_t(t,T) = A(t,T)B(t,T)\left(\alpha(t) - \frac{\gamma(t)}{2}B(t,T)\right), \quad A(T,T) = 1.$$

Remark 4.27 (Zero-coupon price in affine term-structure models). The equation for B does not involve A and is an ordinary differential equation, more precisely, a Riccati differential equation, which should be solved first. Then, using this solution B, the equation for A is another ordinary differential equation, more precisely, a linear differential equation, which can be solved using the formula

$$A(t,T) = \exp\left\{-\int_t^T B(u,T)\left(\alpha(u) - \frac{\gamma(u)}{2}B(u,T)\right)\mathrm{d}u\right\}.$$

Theorem 4.28 (Affine term-structure models). *In an affine term-structure model in which the short rate satisfies the stochastic differential equation*

$$\mathrm{d}r(t) = \mu(t,r(t))\mathrm{d}t + \sigma(t,r(t))\mathrm{d}W(t)$$

with μ and σ as in Theorem 4.26, we have the following formulas:

$$\mathrm{d}P(t,T) = r(t)P(t,T)\mathrm{d}t - \sigma(t,r(t))B(t,T)P(t,T)\mathrm{d}W(t),$$
$$\mathrm{d}\frac{1}{P(t,T)} = \frac{(\sigma^2(t,r(t))B^2(t,T) - r(t)}{P(t,T)}\mathrm{d}t + \frac{\sigma(t,r(t))B(t,T)}{P(t,T)}\mathrm{d}W(t),$$
$$\mathrm{d}W^T(t) = \mathrm{d}W(t) + \sigma(t,r(t))B(t,T)\mathrm{d}t,$$
$$\mathrm{d}r(t) = \left[\mu(t,r(t)) - \sigma^2(t,r(t))B(t,T)\right]\mathrm{d}t + \sigma(t,r(t))\mathrm{d}W^T(t),$$
$$f(t,T) = r(t)B_T(t,T) - \frac{A_T(t,T)}{A(t,T)},$$
$$\mathrm{d}f(t,T) = \sigma(t,r(t))B_T(t,T)\mathrm{d}W^T(t),$$
$$\mathrm{d}F(t;T,S) = \sigma(t,r(t))\left(F(t;T,S) + \frac{1}{\tau(T,S)}\right)(B(t,S) - B(t,T))\mathrm{d}W^S(t).$$