CHAPTER 5

Extended One-Factor Short-Rate Models

5.1. Ho–Le Model

Definition 5.1 (Ho–Le model). In the *Ho–Le model*, the short rate is assumed to satisfy the stochastic differential equation

$$\mathrm{d}r(t) = \theta(t)\mathrm{d}t + \sigma \mathrm{d}W(t),$$

where $\sigma > 0$, θ is deterministic, and W is a Brownian motion under the risk-neutral measure.

Theorem 5.2 (Ho–Le model). *In the Ho–Le model, we have the following formulas:*

$$r(t) = r(s) + \int_s^t \theta(u)\mathrm{d}u + \sigma(W(t) - W(s)),$$

$$\mathbb{E}(r(t)|\mathcal{F}(s)) = r(s) + \int_s^t \theta(u)\mathrm{d}u \quad \textit{and} \quad \mathbb{V}(r(t)|\mathcal{F}(s)) = \sigma^2(t-s),$$

$$P(t,T) = A(t,T)e^{-r(t)(T-t)},$$

$$\textit{where} \quad A(t,T) = \exp\left\{\frac{\sigma^2}{6}(T-t)^3 - \int_t^T (T-u)\theta(u)\mathrm{d}u\right\},$$

$$\mathrm{d}P(t,T) = r(t)P(t,T)\mathrm{d}t - \sigma(T-t)P(t,T)\mathrm{d}W(t),$$

$$\mathrm{d}\frac{1}{P(t,T)} = \frac{\sigma^2(T-t)^2 - r(t)}{P(t,T)}\mathrm{d}t + \frac{\sigma(T-t)}{P(t,T)}\mathrm{d}W(t),$$

$$\mathrm{d}W^T(t) = \mathrm{d}W(t) + \sigma(T-t)\mathrm{d}t,$$

$$\mathrm{d}r(t) = \left[\theta(t) - \sigma^2(T-t)\right]\mathrm{d}t + \sigma \mathrm{d}W^T(t),$$

$$f(t,T) = r(t) - \frac{\sigma^2}{2}(T-t)^2 + \int_t^T \theta(u)\mathrm{d}u \quad \textit{and} \quad \mathrm{d}f(t,T) = \sigma \mathrm{d}W^T(t),$$

$$\mathrm{d}F(t;T,S) = \sigma\left(F(t;T,S) + \frac{1}{\tau(T,S)}\right)(S-T)\mathrm{d}W^S(t),$$

$$\mathrm{ZBC}(t,T,S,K) = P(t,S)\Phi(h) - KP(t,T)\Phi(h-\tilde{\sigma}),$$

$$\mathrm{ZBP}(t,T,S,K) = KP(t,T)\Phi(-h+\tilde{\sigma}) - P(t,S)\Phi(-h),$$

$$\textit{where} \quad \tilde{\sigma} = \sigma(S-T)\sqrt{T-t} \quad \textit{and} \quad h = \frac{1}{\tilde{\sigma}}\ln\left(\frac{P(t,S)}{P(t,T)K}\right) + \frac{\tilde{\sigma}}{2},$$

$$\mathrm{Cap}(t,\mathcal{T},N,K) = N\sum_{i=\alpha+1}^{\beta}\left[P(t,T_{i-1})\Phi(-h_i+\tilde{\sigma}_i) - (1+\tau_i K)P(t,T_i)\Phi(-h_i)\right],$$

$$\mathrm{Flr}(t,\mathcal{T},N,K) = N\sum_{i=\alpha+1}^{\beta}\left[(1+\tau_i K)P(t,T_i)\Phi(h_i) - P(t,T_{i-1})\Phi(h_i-\tilde{\sigma}_i)\right],$$

$$\textit{where} \quad \tilde{\sigma}_i = \sigma(T_i-T_{i-1})\sqrt{T_{i-1}-t} \quad \textit{and} \quad h_i = \frac{1}{\tilde{\sigma}_i}\ln\left(\frac{P(t,T_i)}{P(t,T_{i-1})K}\right) + \frac{\tilde{\sigma}_i}{2}.$$

Theorem 5.3 (Calibration in the Ho–Le model). *If the Ho–Le model is calibrated to a given interest rate structure* $\{f^{\mathrm{M}}(0,t) : t \geq 0\}$*, i.e.,*

$$f(0,t) = f^{\mathrm{M}}(0,t) \quad \textit{for all} \quad t \geq 0,$$

then

$$\theta(t) = \frac{\partial f^{\mathrm{M}}(0,t)}{\partial t} + \sigma^2 t \quad \textit{for all} \quad t \geq 0.$$

Theorem 5.4 (Zero-coupon bond price in the calibrated Ho–Le model). *If the Ho–Le model is calibrated to a given interest rate structure* $\{f^{\mathrm{M}}(0,t) : t \geq 0\}$*, then*

$$P(t,T) = e^{-r(t)(T-t)}\frac{P^{\mathrm{M}}(0,T)}{P^{\mathrm{M}}(0,t)}\exp\left\{(T-t)f^{\mathrm{M}}(0,t) - \frac{\sigma^2}{2}t(T-t)^2\right\},$$

where

$$P^{\mathrm{M}}(0,t) = \exp\left\{-\int_0^t f^{\mathrm{M}}(0,u)\mathrm{d}u\right\} \quad \textit{for all} \quad t \geq 0.$$

5.2. Hull–White Model (Extended Vasicek Model)

Definition 5.5 (Short-rate dynamics in the Hull–White model). In the *Hull–White model*, the short rate is assumed to satisfy the stochastic differential equation

$$\mathrm{d}r(t) = k(\theta(t) - r(t))\mathrm{d}t + \sigma\mathrm{d}W(t),$$

where $k, \sigma > 0$, θ is deterministic, and W is a Brownian motion under the risk-neutral measure.

Remark 5.6 (Hull–White model). The Hull–White model is also called the *extended Vasicek model* or the *G++ model* and can be considered, more generally, with the constants k and σ replaced by deterministic functions.

Theorem 5.7 (Short rate in the Hull–White model). *Let $0 \leq s \leq t \leq T$. The short rate in the Hull–White model is given by*

$$r(t) = r(s)e^{-k(t-s)} + k\int_s^t \theta(u)e^{-k(t-u)}\mathrm{d}u + \sigma\int_s^t e^{-k(t-u)}\mathrm{d}W(u)$$

and is, conditionally on $\mathcal{F}(s)$, normally distributed with

$$\mathbb{E}(r(t)|\mathcal{F}(s)) = r(s)e^{-k(t-s)} + k\int_s^t \theta(u)e^{-k(t-u)}\mathrm{d}u$$

and

$$\mathbb{V}(r(t)|\mathcal{F}(s)) = \frac{\sigma^2}{2k}\left(1 - e^{-2k(t-s)}\right).$$

Remark 5.8 (Short rate in the Hull–White model). As in the Vasicek model, the short rate $r(t)$ in the extended Vasicek model, for each time t, can be negative with positive probability, namely, with probability

$$\Phi\left(-\frac{r(0)e^{-kt} + k\int_0^t \theta(u)e^{-k(t-u)}\mathrm{d}u}{\sqrt{\frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right)}}\right),$$

which is often "negligible in practice". On the other hand, the short rate in the Vasicek model is *mean reverting* provided

$$\varphi^* = \lim_{t\to\infty}\left\{k\int_0^t \theta(u)e^{-k(t-u)}\mathrm{d}u\right\}$$

exists, and then

$$\mathbb{E}(r(t)) \to \varphi^* \quad \text{as} \quad t \to \infty.$$

Theorem 5.9 (Zero-coupon bond in the Hull–White model). *In the Hull–White model, the price of a zero-coupon bond with maturity T at time $t \in [0, T]$ is given by*

$$P(t,T) = \bar{A}(t,T)e^{-r(t)B(t,T)},$$

where

$$\bar{A}(t,T) = A(t,T)\exp\left\{-k\int_t^T \theta(u)B(u,T)\mathrm{d}u\right\}$$

and A and B are as in the Vasicek model, Theorem 4.4 with $\theta = 0$.

THEOREM 5.10 (Forward rate in the Hull–White model). *In the Hull–White model, the instantaneous forward interest rate with maturity T is given by*

$$f(t,T) = k\int_t^T \theta(u)e^{-k(T-u)}\mathrm{d}u - \frac{\sigma^2}{2}B^2(t,T) + r(t)e^{-k(T-t)}.$$

THEOREM 5.11 (Calibration in the Hull–White model). *If the Hull–White model is calibrated to a given interest rate structure $\{f^{\mathrm{M}}(0,t) : t \geq 0\}$, then*

$$\theta(t) = f^{\mathrm{M}}(0,t) + \frac{1}{k}\frac{\partial f^{\mathrm{M}}(0,t)}{\partial t} + \frac{\sigma^2}{2k^2}\left(1-e^{-2kt}\right) \quad \text{for all} \quad t \geq 0.$$

THEOREM 5.12 (Zero-coupon bond in the calibrated Hull–White model). *If the Hull–White model is calibrated to a given interest rate structure, then*

$$P(t,T) = e^{-r(t)B(t,T)}\frac{P^{\mathrm{M}}(0,T)}{P^{\mathrm{M}}(0,t)}\exp\left\{B(t,T)f^{\mathrm{M}}(0,t) - \frac{\sigma^2}{4k}\left(1-e^{-2kt}\right)B^2(t,T)\right\}.$$

THEOREM 5.13 (Option on a zero-coupon bond in the Hull–White model). *In the Hull–White model, the price of a European call option with strike K and maturity T and written on a zero-coupon bond with maturity S at time $t \in [0,T]$ is given by*

$$\mathrm{ZBC}(t,T,S,K) = P(t,S)\Phi(h) - KP(t,T)\Phi(h-\tilde{\sigma}),$$

where $\tilde{\sigma}$ and h are as in the Vasicek model, Theorem 4.9.

$$\mathrm{ZBP}(t,T,S,K) = KP(t,T)\Phi(-h+\tilde{\sigma}) - P(t,S)\Phi(-h).$$

THEOREM 5.14 (Caps and floors in the Hull–White model). *In the Hull–White model, the price of a cap with notional value N, cap rate K, and the set of times $\mathcal{T}$, is given by*

$$\mathrm{Cap}(t,\mathcal{T},N,K) = N\sum_{i=\alpha+1}^{\beta}\left[P(t,T_{i-1})\Phi(-h_i+\tilde{\sigma}_i) - (1+\tau_i K)P(t,T_i)\Phi(-h_i)\right],$$

while the price of a floor with notional value N, floor rate K, and the set of times $\mathcal{T}$, is given by

$$\mathrm{Flr}(t,\mathcal{T},N,K) = N\sum_{i=\alpha+1}^{\beta}\left[(1+\tau_i K)P(t,T_i)\Phi(h_i) - P(t,T_{i-1})\Phi(h_i-\tilde{\sigma}_i)\right],$$

where $\tilde{\sigma}_i$ and h_i are as in the Vasicek model, Theorem 4.10.

5.3. Black–Karasinski Model

DEFINITION 5.15 (Black–Karasinski model). In the *Black–Karasinski model*, the short rate is given by

$$r(t) = e^{y(t)} \quad \text{with} \quad \mathrm{d}y(t) = k(\theta(t) - y(t))\mathrm{d}t + \sigma \mathrm{d}W(t),$$

where $k, \sigma > 0$, θ is deterministic, and W is a Brownian motion under the risk-neutral measure.

REMARK 5.16 (Black–Karasinski model). The Black–Karasinski model is also called the *extended exponential Vasicek model* and can be considered, more generally, with the constants k and σ replaced by deterministic functions.

THEOREM 5.17 (Short rate in the Black–Karasinski model). *The short rate in the Black–Karasinski model satisfies the stochastic differential equation*

$$\mathrm{d}r(t) = \left(k\theta(t) + \frac{\sigma^2}{2} - k\ln(r(t))\right) r(t)\mathrm{d}t + \sigma r(t)\mathrm{d}W(t).$$

Let $0 \le s \le t \le T$. *Then* r *is given by*

$$r(t) = \exp\left\{\ln(r(s))e^{-k(t-s)} + k\int_s^t e^{-k(t-u)}\theta(u)\mathrm{d}u + \sigma\int_s^t e^{-k(t-u)}\mathrm{d}W(u)\right\}$$

and is, conditionally on $\mathcal{F}(s)$, *lognormally distributed with*

$$\begin{aligned}&\mathbb{E}(r(t)|\mathcal{F}(s)) \\ &\qquad = \exp\left\{\ln(r(s))e^{-k(t-s)} + k\int_s^t e^{-k(t-u)}\theta(u)\mathrm{d}u + \frac{\sigma^2}{4k}\left(1 - e^{-2k(t-s)}\right)\right\}\end{aligned}$$

and

$$\begin{aligned}\mathbb{V}(r(t)|\mathcal{F}(s)) &= \exp\left\{2\ln(r(s))e^{-k(t-s)} + 2k\int_s^t e^{-k(t-u)}\theta(u)\mathrm{d}u\right\} \times \\ &\quad \times \exp\left\{\frac{\sigma^2}{2k}\left(1 - e^{-2k(t-s)}\right)\right\}\left[\exp\left\{\frac{\sigma^2}{2k}\left(1 - e^{-2k(t-s)}\right)\right\} - 1\right].\end{aligned}$$

REMARK 5.18 (Short rate in the Black–Karasinski model). Since the short rate r in the Black–Karasinski model is lognormally distributed, it is always positive. A disadvantage is that $P(t, T)$ cannot be calculated explicitly. An advantage of the Black–Karasinski model is that r is always mean reverting provided

$$\varphi^* = \lim_{t\to\infty}\left\{k\int_0^t \theta(u)e^{-k(t-u)}\mathrm{d}u\right\}$$

exists, and then

$$\mathbb{E}(r(t)|\mathcal{F}(s)) \to \exp\left(\varphi^* + \frac{\sigma^2}{4k}\right) \quad \text{as} \quad t \to \infty$$

and

$$\mathbb{V}(r(t)|\mathcal{F}(s)) \to \exp\left(2\varphi^* + \frac{\sigma^2}{2k}\right)\left[\exp\left(\frac{\sigma^2}{2k}\right) - 1\right] \quad \text{as} \quad t \to \infty.$$

5.4. Deterministic-Shift Extended Models

DEFINITION 5.19 (Short rate in a deterministic-shift extended model). In a *deterministic-shift extended model*, the short rate is given by

$$r(t) = x(t) + \varphi(t) \quad \text{with} \quad \mathrm{d}x(t) = \mu(t, x(t))\mathrm{d}t + \sigma(t, x(t))\mathrm{d}W(t),$$

where φ, μ, σ are deterministic functions and W is a Brownian motion under the risk-neutral measure. The stochastic differential equation for x is called the *reference model*, and prices of zero-coupon bonds and forward interest rates in the reference model are denoted by $P_x^{\mathrm{REF}}(t,T)$ and $f_x^{\mathrm{REF}}(t,T)$, respectively.

THEOREM 5.20 (Zero-coupon bond in a deterministic-shift extended model). *In a deterministic-shift extended model, the price of a zero-coupon bond with maturity T at time $t \in [0,T]$ is given by*

$$P(t,T) = \exp\left(-\int_t^T \varphi(u)\mathrm{d}u\right) P_{r-\varphi}^{\mathrm{REF}}(t,T).$$

THEOREM 5.21 (Forward rate in a deterministic-shift extended model). *In a deterministic-shift extended model, the instantaneous forward interest rate with maturity T is given by*

$$f(t,T) = \varphi(T) + f_{r-\varphi}^{\mathrm{REF}}(t,T).$$

THEOREM 5.22 (Calibration in a deterministic-shift extended model). *If a deterministic-shift extended model is calibrated to a given interest rate structure $\{f^{\mathrm{M}}(0,t) : t \geq 0\}$, then*

$$\varphi(t) = f^{\mathrm{M}}(0,t) - f_{r-\varphi}^{\mathrm{REF}}(0,t) \quad \textit{for all} \quad t \geq 0.$$

THEOREM 5.23 (Zero-coupon bond in a calibrated deterministic-shift extended model). *If a deterministic-shift extended model is calibrated to a given interest rate structure, then*

$$P(t,T) = \frac{P^{\mathrm{M}}(0,T)}{P^{\mathrm{M}}(0,t)} \frac{P^{\mathrm{REF}}_{r-\varphi}(0,t)}{P^{\mathrm{REF}}_{r-\varphi}(0,T)} P^{\mathrm{REF}}_{r-\varphi}(t,T).$$

THEOREM 5.24 (Option on a zero-coupon bond in a deterministic-shift extended model). *In a deterministic-shift extended model, the price of a European call option with strike K and maturity T and written on a zero-coupon bond with maturity S at time $t \in [0,T]$ is given by*

$$\mathrm{ZBC}(t,T,S,K) = \exp\left(-\int_t^S \varphi(u)\mathrm{d}u\right) \mathrm{ZBC}^{\mathrm{REF}}_{r-\varphi}(t,T,S,K'),$$

where

$$K' = K \exp\left(\int_T^S \varphi(u)\mathrm{d}u\right).$$

5.5. Extended CIR Model

DEFINITION 5.25 (Short rate in the extended CIR model). In the *extended CIR model*, the short rate is given by

$$r(t) = x(t) + \varphi(t) \quad \text{with} \quad \mathrm{d}x(t) = k(\theta - x(t))\mathrm{d}t + \sigma\sqrt{x(t)}\mathrm{d}W(t),$$

where $k, \sigma, \theta > 0$ and W is a Brownian motion under the risk-neutral measure.

REMARK 5.26 (Extended CIR model). The extended CIR model is also called the *CIR++ model* and can be considered, more generally, with the constants k and σ replaced by deterministic functions.

THEOREM 5.27 (Zero-coupon bond in the CIR++ model). *In the CIR++ model, the price of a zero-coupon bond with maturity T at time $t \in [0,T]$ is given by*

$$P(t,T) = \bar{A}(t,T)e^{-r(t)B(t,T)},$$

where

$$\bar{A}(t,T) = A(t,T)\exp\left\{\varphi(t)B(t,T) - \int_t^T \varphi(u)\mathrm{d}u\right\}$$

and A and B are as in the CIR model, Theorem 4.20.

THEOREM 5.28 (Forward rate in the CIR++ model). *In the CIR++ model, the instantaneous forward interest rate with maturity T is given by*

$$f(t,T) = \varphi(T) - \varphi(t)B_T(t,T) + k\theta B(t,T) + r(t)B_T(t,T),$$

where B is as in the CIR model, Theorem 4.20.

THEOREM 5.29 (Calibration in the CIR++ model). *If the CIR++ model is calibrated to a given interest rate structure $\{f^{\mathrm{M}}(0,t) : t \geq 0\}$, then*

$$\varphi(t) = f^{\mathrm{M}}(0,t) + \varphi(0)B_T(0,T) - k\theta B(0,T) - r(0)B_T(0,T) \quad \textit{for all} \quad t \geq 0,$$

where B is as in the CIR model, Theorem 4.20.

THEOREM 5.30 (Zero-coupon bond in the CIR++ model). *If the CIR++ model is calibrated to a given interest rate structure, then*

$$P(t,T) = \frac{P^{\mathrm{M}}(0,T)}{P^{\mathrm{M}}(0,t)} \frac{A(0,t)A(t,T)}{A(0,T)} e^{(r(0)-\varphi(0))(B(0,T)-B(0,t))+\varphi(t)B(t,T)} e^{-r(t)B(t,T)},$$

where A and B are as in the CIR model, Theorem 4.20.

5.6. Extended Affine Term-Structure Models

THEOREM 5.31 (Extended affine term-structure models). *Assume the reference model is an affine term-structure model, i.e.,*

$$P_r^{\mathrm{REF}}(t,T) = A(t,T)e^{-r(t)B(t,T)}.$$

If this model is extended according to Definition 5.19 by using the deterministic shift φ, then we have the following formulas:

$$P(t,T) = \bar{A}(t,T)e^{-r(t)B(t,T)},$$

$$\textit{where} \quad \bar{A}(t,T) = A(t,T)\exp\left\{\varphi(t)B(t,T) - \int_t^T \varphi(u)\mathrm{d}u\right\},$$

$$f(t,T) = \varphi(T) - \varphi(t)B_T(t,T) - \frac{A_T(t,T)}{A(t,T)} + r(t)B_T(t,T),$$

and if the extended model is calibrated to a given interest rate structure, then

$$\varphi(t) = f^{\mathrm{M}}(0,t) + (\varphi(0) - r(0))B_T(0,t) + \frac{A_T(0,t)}{A(0,t)},$$

$$P(t,T) = \frac{P^{\mathrm{M}}(0,T)}{P^{\mathrm{M}}(0,t)} \frac{A(0,t)A(t,T)}{A(0,T)} e^{(r(0)-\varphi(0))(B(0,T)-B(0,t))+\varphi(t)B(t,T)} e^{-r(t)B(t,T)}.$$