CHAPTER 6

Two-Factor Short-Rate Models

6.1. G2++ Model

Remark 6.1 (Motivation). In an affine term-structure model, $f(t, T_1)$ and $f(t, T_2)$ with $T_1 = t+1$ and $T_2 = t+100$ ("short" and "long" rate) are perfectly correlated, i.e., their correlation coefficient is one, which is not realistic.

Definition 6.2 (Short-rate dynamics in the G2++ model). In the *G2++ model*, the short rate is given by

$$r(t) = x_1(t) + x_2(t) + \varphi(t),$$

where φ is deterministic and x_1 and x_2 are assumed to satisfy the stochastic problems

$$\mathrm{d}x_1(t) = -k_1 x_1(t)\mathrm{d}t + \sigma_1 \mathrm{d}W_1(t), \quad x_1(0) = 0$$

and

$$\mathrm{d}x_2(t) = -k_2 x_2(t)\mathrm{d}t + \sigma_2 \mathrm{d}W_2(t), \quad x_2(0) = 0,$$

where $k_1, k_2, \sigma_1, \sigma_2 > 0$ and W_1 and W_2 are Brownian motions under the risk-neutral measure such that

$$\mathrm{d}W_1(t)\mathrm{d}W_2(t) = \rho \mathrm{d}t \quad \text{with} \quad \rho \in [-1, 1].$$

Theorem 6.3 (Short rate in the G2++ model). *Let $0 \le s \le t \le T$. The short rate in the G2++ model is given by*

$$\begin{aligned} r(t) &= x_1(s)e^{-k_1(t-s)} + x_2(s)e^{-k_2(t-s)} + \varphi(t) \\ &\quad + \sigma_1 \int_s^t e^{-k_1(t-u)}\mathrm{d}W_1(u) + \sigma_2 \int_s^t e^{-k_2(t-u)}\mathrm{d}W_2(u) \end{aligned}$$

and is, conditionally on $\mathcal{F}(s)$, normally distributed with

$$\mathbb{E}(r(t)|\mathcal{F}(s)) = x_1(s)e^{-k_1(t-s)} + x_2(s)e^{-k_2(t-s)} + \varphi(t)$$

and

$$\begin{aligned}\mathbb{V}(r(t)|\mathcal{F}(s)) &= \frac{\sigma_1^2}{2k_1}\left(1-e^{-2k_1(t-s)}\right)+\frac{\sigma_2^2}{2k_2}\left(1-e^{-2k_2(t-s)}\right)\\ &\quad+\frac{2\sigma_1\sigma_2\rho}{k_1+k_2}\left(1-e^{-(k_1+k_2)(t-s)}\right).\end{aligned}$$

THEOREM 6.4 (Zero-coupon bond in the G2++ model). *In the G2++ model, the price of a zero-coupon bond with maturity T at time $t \in [0,T]$ is given by*

$$P(t,T)=\exp\left\{-\int_t^T \varphi(u)\mathrm{d}u - M(t,T)+\frac{1}{2}V^2(t,T)\right\},$$

where

$$M(t,T)=x_1(t)B_1(t,T)+x_2(t)B_2(t,T)$$

and

$$\begin{aligned}V^2(t,T) &= \frac{\sigma_1^2}{k_1^2}\left(T-t-B_1(t,T)-\frac{k_1}{2}B_1^2(t,T)\right)\\ &\quad+\frac{\sigma_2^2}{k_2^2}\left(T-t-B_2(t,T)-\frac{k_2}{2}B_2^2(t,T)\right)\\ &\quad+\frac{2\sigma_1\sigma_2\rho}{k_1k_2}\left(T-t-B_1(t,T)-B_2(t,T)+B_{12}(t,T)\right),\end{aligned}$$

where

$$B_1(t,T)=\frac{1-e^{-k_1(T-t)}}{k_1},\quad B_2(t,T)=\frac{1-e^{-k_2(T-t)}}{k_2},$$

and

$$B_{12}(t,T)=\frac{1-e^{-(k_1+k_2)(T-t)}}{k_1+k_2}.$$

THEOREM 6.5 (Forward rate in the G2++ model). *In the G2++ model, the instantaneous forward rate with maturity T is given by*

$$\begin{aligned}f(t,T) &= \varphi(T)+x_1(t)e^{-k_1(T-t)}+x_2(t)e^{-k_2(T-t)}\\ &\quad-\frac{\sigma_1^2}{2}B_1^2(t,T)-\frac{\sigma_2^2}{2}B_2^2(t,T)-\sigma_1\sigma_2\rho B_1(t,T)B_2(t,T).\end{aligned}$$

THEOREM 6.6 (Calibration in the G2++ model). *If the CIR2++ model is calibrated to a given interest rate structure $\{f^{\mathrm{M}}(0,t): t\geq 0\}$, then*

$$\varphi(t)=f^{\mathrm{M}}(0,t)+\frac{\sigma_1^2}{2}B_1^2(0,t)+\frac{\sigma_2^2}{2}B_2^2(0,t)+\sigma_1\sigma_2\rho B_1(0,t)B_2(0,t).$$

THEOREM 6.7 (Zero-coupon bond in the calibrated G2++ model). *If the G2++ model is calibrated to a given interest rate structure, then*

$$P(t,T) = \frac{P^{\mathrm{M}}(0,T)}{P^{\mathrm{M}}(0,t)} \exp\left\{\frac{1}{2}\left(V^2(t,T) - V^2(0,T) + V^2(0,t)\right) - M(t,T)\right\},$$

where M and V^2 are given in Theorem 6.4.

THEOREM 6.8 (Bond-price dynamics in the G2++ model). *In the G2++ model, the price of a zero-coupon bond with maturity T satisfies the stochastic differential equations*

$$\mathrm{d}P(t,T) = r(t)P(t,T)\mathrm{d}t - \sigma_1 B_1(t,T)P(t,T)\mathrm{d}W_1(t) - \sigma_2 B_2(t,T)P(t,T)\mathrm{d}W_2(t)$$

and

$$\begin{aligned}\mathrm{d}\frac{1}{P(t,T)} \quad &= \quad \frac{\sigma_1^2 B_1^2(t,T) + \sigma_2^2 B_2^2(t,T) + 2\sigma_1\sigma_2\rho B_1(t,T)B_2(t,T) - r(t)}{P(t,T)}\mathrm{d}t \\ &\quad + \frac{\sigma_1 B_1(t,T)}{P(t,T)}\mathrm{d}W_1(t) + \frac{\sigma_2 B_2(t,T)}{P(t,T)}\mathrm{d}W_2(t).\end{aligned}$$

THEOREM 6.9 (T-forward measure dynamics of the short rate in the G2++ model). *Under the T-forward measure $\mathbb{Q}^T$, the short rate r in the G2++ model satisfies $r(t) = x_1(t)+x_2(t)+\varphi(t)$ such that x_1 and x_2 are solutions of the stochastic differential equations*

$$\mathrm{d}x_1(t) = -\left(\sigma_1^2 B_1(t,T) + \sigma_1\sigma_2\rho B_2(t,T) + k_1 x_1(t)\right)\mathrm{d}t + \sigma_1 \mathrm{d}W_1^T(t)$$

and

$$\mathrm{d}x_2(t) = -\left(\sigma_2^2 B_2(t,T) + \sigma_1\sigma_2\rho B_1(t,T) + k_2 x_2(t)\right)\mathrm{d}t + \sigma_2 \mathrm{d}W_2^T(t),$$

where the $\mathbb{Q}^T$-Brownian motions W_1^T and W_2^T are defined by

$$\mathrm{d}W_1^T(t) = \mathrm{d}W_1(t) + \left(\sigma_1 B_1(t,T) + \sigma_2\rho B_2(t,T)\right)\mathrm{d}t$$

and

$$\mathrm{d}W_2^T(t) = \mathrm{d}W_2(t) + \left(\sigma_2 B_2(t,T) + \sigma_1\rho B_1(t,T)\right)\mathrm{d}t.$$

THEOREM 6.10 (Forward-rate dynamics in the G2++ model). *In the G2++ model, the instantaneous forward interest rate with maturity T satisfies the stochastic differential equation*

$$\mathrm{d}f(t,T) = \sigma_1 e^{-k_1(T-t)}\mathrm{d}W_1^T(t) + \sigma_2 e^{-k_2(T-t)}\mathrm{d}W_2^T(t).$$

Theorem 6.11 (Forward-rate dynamics in the G2++ model). *In the G2++ model, the simply-compounded forward interest rate for the period $[T, S]$ satisfies the stochastic differential equation*

$$\begin{aligned} \mathrm{d}F(t;T,S) &= \sigma_1 \left(F(t;T,S) + \frac{1}{\tau(T,S)} \right) (B_1(t,S) - B_1(t,T)) \mathrm{d}W_1^S(t) \\ &\quad + \sigma_2 \left(F(t;T,S) + \frac{1}{\tau(T,S)} \right) (B_2(t,S) - B_2(t,T)) \mathrm{d}W_2^S(t). \end{aligned}$$

Theorem 6.12 (Option on a zero-coupon bond in the G2++ model). *In the G2++ model, the price of a European call option with strike K and maturity T and written on a zero-coupon bond with maturity S at time $t \in [0, T]$ is given by*

$$\mathrm{ZBC}(t,T,S,K) = P(t,S)\Phi(h) - KP(t,T)\Phi(h - \hat{\sigma}),$$

where

$$\begin{aligned} \hat{\sigma}^2 &= \frac{\sigma_1^2}{2k_1} \left(1 - e^{-2k_1(T-t)} \right) B_1^2(T,S) + \frac{\sigma_2^2}{2k_2} \left(1 - e^{-2k_2(T-t)} \right) B_2^2(T,S) \\ &\quad + 2\sigma_1\sigma_2\rho B_1(T,S) B_2(T,S) B_{12}(t,T) \end{aligned}$$

and

$$h = \frac{1}{\hat{\sigma}} \ln \left(\frac{P(t,S)}{P(t,T)K} \right) + \frac{\hat{\sigma}}{2}.$$

The price of a corresponding put option is given by

$$\mathrm{ZBP}(t,T,S,K) = KP(t,T)\Phi(-h + \hat{\sigma}) - P(t,S)\Phi(-h).$$

Theorem 6.13 (Caps and floors in the G2++ model). *In the G2++ model, the price of a cap with notional value N, cap rate K, and the set of times $\mathcal{T}$, is given by*

$$\mathrm{Cap}(t,\mathcal{T},N,K) = N \sum_{i=\alpha+1}^{\beta} \left[P(t,T_{i-1})\Phi(-h_i + \hat{\sigma}_i) - (1 + \tau_i K) P(t,T_i)\Phi(-h_i) \right],$$

while the price of a floor with notional value N, floor rate K, and the set of times $\mathcal{T}$, is given by

$$\mathrm{Flr}(t,\mathcal{T},N,K) = N \sum_{i=\alpha+1}^{\beta} \left[(1 + \tau_i K) P(t,T_i)\Phi(h_i) - P(t,T_{i-1})\Phi(h_i - \hat{\sigma}_i) \right],$$

where

$$\begin{aligned}\hat{\sigma}^2 &= \frac{\sigma_1^2}{2k_1}\left(1-e^{-2k_1(T_{i-1}-t)}\right)B_1^2(T_{i-1},T_i)\\ &\quad+\frac{\sigma_2^2}{2k_2}\left(1-e^{-2k_2(T_{i-1}-t)}\right)B_2^2(T_{i-1},T_i)\\ &\quad+2\sigma_1\sigma_2\rho B_1(T_{i-1},T_i)B_2(T_{i-1},T_i)B_{12}(t,T_{i-1})\end{aligned}$$

and

$$h_i = \frac{1}{\hat{\sigma}_i}\ln\left(\frac{(1+\tau_i K)P(t,T_i)}{P(t,T_{i-1})}\right)+\frac{\hat{\sigma}_i}{2}.$$

THEOREM 6.14 (Swaptions in the G2++ model). *In the G2++ model, the price at time 0 of a European payer swaption with swaption maturity $T = T_\alpha$ on an IRS depending on the notional value N, the fixed rate K, and the set of times $\mathcal{T} = \{T_{\alpha+1},\dots,T_\beta\}$ is given by numerically computing the one-dimensional integral*

$$\begin{aligned}&\mathrm{PS}(0,T,\mathcal{T},N,K)\\ &\quad= NP(0,T)\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-\tilde{\mu}_1)^2}{2\tilde{\sigma}_1^2}}}{\tilde{\sigma}_1\sqrt{2\pi}}\left[\Phi(-h_1(x))-\sum_{i=\alpha+1}^{\beta}\lambda_i(x)e^{\kappa_i(x)}\Phi(-h_2(x))\right]\mathrm{d}x,\end{aligned}$$

where

$$h_1(x) = \frac{\bar{x}-\tilde{\mu}_2}{\tilde{\sigma}_2\sqrt{1-\tilde{\rho}^2}}-\frac{\tilde{\rho}(x-\tilde{\mu}_1)}{\tilde{\sigma}_1\sqrt{1-\tilde{\rho}^2}},\quad h_2(x) = h_1(x)+B_2(T,T_i)\tilde{\sigma}_2\sqrt{1-\tilde{\rho}^2},$$

$$\lambda_i(x) = c_i A(T,T_i)e^{-xB_1(T,T_i)},\qquad \sum_{i=\alpha+1}^{\beta}\lambda_i e^{-\bar{x}B_2(T,T_i)} = 1,$$

$$c_i = K\tau_i \quad \textit{for} \quad \alpha < i < \beta \quad \textit{and} \quad c_\beta = 1+K\tau_\beta,$$

$$\kappa_i(x) = -B_2(T,T_i)\left[\tilde{\mu}_2-\frac{\tilde{\sigma}_2^2(1-\tilde{\rho}^2)}{2}B_2(T,T_i)+\tilde{\rho}\tilde{\sigma}_2\frac{x-\tilde{\mu}_1}{\tilde{\sigma}_1}\right],$$

$$\tilde{\mu}_1 = \frac{\sigma_1^2}{2k_1^2}(1-e^{-2k_1T})+\frac{\sigma_1\sigma_2\rho}{k_2}B_{12}(0,T)-\left(\frac{\sigma_1^2}{k_1}+\frac{\sigma_1\sigma_2\rho}{k_2}\right)B_1(0,T),$$

$$\tilde{\mu}_2 = \frac{\sigma_2^2}{2k_2^2}(1-e^{-2k_2T})+\frac{\sigma_1\sigma_2\rho}{k_1}B_{12}(0,T)-\left(\frac{\sigma_2^2}{k_2}+\frac{\sigma_1\sigma_2\rho}{k_1}\right)B_2(0,T),$$

$$\tilde{\sigma}_1 = \sigma_1\sqrt{\frac{1-e^{-2k_1T}}{2k_1}},\quad \tilde{\sigma}_2 = \sigma_2\sqrt{\frac{1-e^{-2k_2T}}{2k_2}},\quad \tilde{\rho} = \frac{\sigma_1\sigma_2\rho}{\tilde{\sigma}_1\tilde{\sigma}_2}B_{12}(0,T),$$

$$A(T,T_i) = \exp\left\{\frac{1}{2}V^2(T,T_i)-\int_T^{T_i}\varphi(u)\mathrm{d}u\right\}.$$

6.2. Hull–White Two-Factor Model

DEFINITION 6.15 (Short-rate dynamics in the Hull–White two-factor model). In the *Hull–White two-factor model*, the short rate is assumed to satisfy the stochastic differential equation

$$\mathrm{d}r(t) = k_1(\theta(t) + y(t) - r(t))\mathrm{d}t + \bar{\sigma}_1 \mathrm{d}\bar{W}_1(t),$$

where

$$\mathrm{d}y(t) = -k_2 y(t)\mathrm{d}t + \bar{\sigma}_2 \mathrm{d}\bar{W}_2(t), \quad y(0) = 0,$$

$k_1, k_2, \bar{\sigma}_1, \bar{\sigma}_2 > 0$ and $\bar{W}_1$ and $\bar{W}_2$ are Brownian motions under the risk-neutral measure such that

$$\mathrm{d}\bar{W}_1(t)\mathrm{d}\bar{W}_2(t) = \bar{\rho}\mathrm{d}t \quad \text{with} \quad \bar{\rho} \in [-1, 1].$$

THEOREM 6.16 (Short rate in the Hull–White two-factor model). *Let $k_1 \neq k_2$. Let $0 \leq s \leq t \leq T$. The short rate in the Hull–White two-factor model is given by*

$$\begin{aligned} r(t) &= r(s)e^{-k_1(t-s)} + k_1 \int_s^t \theta(u)e^{-k_1(t-u)}\mathrm{d}u + \bar{\sigma}_1 \int_s^t e^{-k_1(t-u)}\mathrm{d}\bar{W}_1(u) \\ &\quad + k_1\bar{\sigma}_2 \int_s^t \frac{e^{-k_2(t-u)} - e^{-k_1(t-u)}}{k_1 - k_2}\mathrm{d}\bar{W}_2(u) + k_1 y(s)\frac{e^{-k_2(t-s)} - e^{-k_1(t-s)}}{k_1 - k_2}. \end{aligned}$$

In particular, we have

$$\begin{aligned} r(t) &= r(0)e^{-k_1 t} + k_1 \int_0^t \theta(u)e^{-k_1(t-u)}\mathrm{d}u + \frac{k_1\bar{\sigma}_2}{k_1 - k_2} \int_0^t e^{-k_2(t-u)}\mathrm{d}\bar{W}_2(u) \\ &\quad + \int_0^t e^{-k_1(t-u)} \left\{ \bar{\sigma}_1 \mathrm{d}\bar{W}_1(u) - \frac{k_1\bar{\sigma}_2}{k_1 - k_2}\mathrm{d}\bar{W}_2(u) \right\}. \end{aligned}$$

THEOREM 6.17 (The Hull–White two-factor model and the G2++ model). *Suppose r is the short rate in the Hull–White two-factor model. Assume $k_1 > k_2$ and define*

$$\varphi(t) = r(0)e^{-k_1 t} + k_1 \int_0^t \theta(u)e^{-k_1(t-u)}\mathrm{d}u,$$

$$\sigma_2 = \frac{k_1\bar{\sigma}_2}{k_1 - k_2}, \quad \sigma_1 = \sqrt{\bar{\sigma}_1^2 - 2\bar{\sigma}_1\sigma_2\bar{\rho} + \sigma_2^2}, \quad \rho = \frac{\bar{\sigma}_1\bar{\rho} - \sigma_2}{\sigma_1},$$

$$W_1 = \frac{\bar{\sigma}_1\bar{W}_1 - \sigma_2\bar{W}_2}{\sigma_1}, \quad W_2 = \bar{W}_2.$$

Then r is equal to the short rate in the corresponding G2++ model.

REMARK 6.18 (The Hull–White two-factor model and the G2++ model). If we assume $k_1 < k_2$ in Theorem 6.17, then we can obtain a similar result. Moreover, it is also possible to recover the short rate in the Hull–White two-factor model from the short rate of the G2++ model.

6.3. CIR2 Model

DEFINITION 6.19 (Short-rate dynamics in the CIR2 model). In the *CIR2 model*, the short rate is given by

$$r(t) = x_1(t) + x_2(t),$$

where x_1 and x_2 are assumed to satisfy the stochastic differential equations

$$\mathrm{d}x_1(t) = k_1(\theta_1 - x_1(t))\mathrm{d}t + \sigma_1\sqrt{x_1(t)}\mathrm{d}W_1(t)$$

and

$$\mathrm{d}x_2(t) = k_2(\theta_2 - x_2(t))\mathrm{d}t + \sigma_2\sqrt{x_2(t)}\mathrm{d}W_2(t),$$

where $k_1, k_2, \theta_1, \theta_2, \sigma_1, \sigma_2 > 0$ such that $2k_1\theta_1 > \sigma_1^2$ and $2k_2\theta_2 > \sigma_2^2$ and W_1 and W_2 are independent Brownian motions under the risk-neutral measure.

THEOREM 6.20 (Zero-coupon bond in the CIR2 model). *In the CIR2 model, the price of a zero-coupon bond with maturity T at time $t \in [0,T]$ is given by*

$$P(t,T) = A_1(t,T)A_2(t,T)e^{-x_1(t)B_1(t,T)-x_2(t)B_2(t,T)},$$

where A_i and B_i for $i \in \{1,2\}$ are as in Theorem 4.20 with k, θ, and σ replaced by k_i, θ_1, and σ_i, respectively.

THEOREM 6.21 (Bond-price dynamics in the CIR2 model). *In the CIR2 model, the price of a zero-coupon bond with maturity T satisfies the stochastic differential equations*

$$\begin{aligned}\mathrm{d}P(t,T) &= r(t)P(t,T)\mathrm{d}t - \sigma_1\sqrt{x_1(t)}B_1(t,T)P(t,T)\mathrm{d}W_1(t) \\ &\quad -\sigma_2\sqrt{x_2(t)}B_2(t,T)P(t,T)\mathrm{d}W_2(t)\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\frac{1}{P(t,T)} &= \frac{(\sigma_1^2B_1^2(t,T)-1)x_1(t) + (\sigma_2^2B_2^2(t,T)-1)x_2(t)}{P(t,T)}\mathrm{d}t \\ &\quad + \frac{\sigma_1\sqrt{x_1(t)}B_1(t,T)}{P(t,T)}\mathrm{d}W_1(t) + \frac{\sigma_2\sqrt{x_2(t)}B_2(t,T)}{P(t,T)}\mathrm{d}W_2(t).\end{aligned}$$

THEOREM 6.22 (T-forward measure dynamics of the short rate in the CIR2 model). *Under the T-forward measure $\mathbb{Q}^T$, the short rate r in the CIR2 model satisfies $r(t) = x_1(t) + x_2(t)$ such that x_1 and x_2 are solutions of the stochastic differential equations*

$$\mathrm{d}x_1(t) = \left[k_1\theta_1 - (k_1 + \sigma_1^2 B_1(t,T))x_1(t)\right]\mathrm{d}t + \sigma_1\sqrt{x_1(t)}\mathrm{d}W_1^T(t)$$

and

$$\mathrm{d}x_2(t) = \left[k_2\theta_2 - (k_2 + \sigma_2^2 B_2(t,T))x_2(t)\right]\mathrm{d}t + \sigma_2\sqrt{x_2(t)}\mathrm{d}W_2^T(t),$$

where the $\mathbb{Q}^T$-Brownian motions W_1^T and W_2^T are defined by

$$\mathrm{d}W_1^T(t) = \mathrm{d}W_1(t) + \sigma_1\sqrt{x_1(t)}B_1(t,T)$$

and

$$\mathrm{d}W_2^T(t) = \mathrm{d}W_2(t) + \sigma_2\sqrt{x_2(t)}B_2(t,T).$$

THEOREM 6.23 (Forward-rate dynamics in the CIR2 model). *In the CIR2 model, the instantaneous forward interest rate with maturity T is given by*

$$f(t,T) = k_1\theta_1 B_1(t,T) + k_2\theta_2 B_2(t,T) + x_1(t)\frac{\partial}{\partial T}B_1(t,T) + x_2(t)\frac{\partial}{\partial T}B_2(t,T)$$

and satisfies the stochastic differential equation

$$\mathrm{d}f(t,T) = \sigma_1\sqrt{x_1(t)}\frac{\partial}{\partial T}B_1(t,T)\mathrm{d}W_1^T(t) + \sigma_2\sqrt{x_2(t)}\frac{\partial}{\partial T}B_2(t,T)\mathrm{d}W_2^T(t).$$

6.4. Longstaff–Schwartz Model

DEFINITION 6.24 (Short-rate dynamics in the Longstaff–Schwartz model). In the *Longstaff–Schwartz model*, the short rate is given by

$$r(t) = \sigma_1^2\bar{x}_1(t) + \sigma_2^2\bar{x}_2(t),$$

where $\bar{x}_1$ and $\bar{x}_2$ are assumed to satisfy the stochastic differential equations

$$\mathrm{d}\bar{x}_1(t) = k_1(\bar{\theta}_1 - \bar{x}_1(t))\mathrm{d}t + \sqrt{\bar{x}_1(t)}\mathrm{d}W_1(t)$$

and

$$\mathrm{d}\bar{x}_2(t) = k_2(\bar{\theta}_2 - \bar{x}_2(t))\mathrm{d}t + \sqrt{\bar{x}_2(t)}\mathrm{d}W_2(t),$$

where $k_1, k_2, \bar{\theta}_1, \bar{\theta}_2, \sigma_1, \sigma_2 > 0$ such that $2k_1\bar{\theta}_1 > 1$ and $2k_2\bar{\theta}_2 > 1$ and W_1 and W_2 are independent Brownian motions under the risk-neutral measure.

THEOREM 6.25 (The Longstaff–Schwartz model and the CIR2 model). *Suppose r is the short rate in the Longstaff–Schwartz model. Define*

$$\theta_1 = \sigma_1^2 \bar{\theta}_1 \quad \text{and} \quad \theta_2 = \sigma_2^2 \bar{\theta}_2.$$

Then r is equal to the short rate in the corresponding CIR2 model.

6.5. CIR2++ Model

DEFINITION 6.26 (Short-rate dynamics in the CIR2++ model). In the *CIR2++ model*, the short rate is given by

$$r(t) = x_1(t) + x_2(t) + \varphi(t),$$

where φ is deterministic and x_1 and x_2 are assumed to satisfy the stochastic differential equations

$$\mathrm{d}x_1(t) = k_1(\theta_1 - x_1(t))\mathrm{d}t + \sigma_1 \sqrt{x_1(t)}\mathrm{d}W_1(t)$$

and

$$\mathrm{d}x_2(t) = k_2(\theta_2 - x_2(t))\mathrm{d}t + \sigma_2 \sqrt{x_2(t)}\mathrm{d}W_2(t),$$

where $k_1, k_2, \theta_1, \theta_2, \sigma_1, \sigma_2 > 0$ such that $2k_1\theta_1 > \sigma_1^2$ and $2k_2\theta_2 > \sigma_2^2$ and W_1 and W_2 are independent Brownian motions under the risk-neutral measure.

THEOREM 6.27 (Zero-coupon bond in the CIR2++ model). *In the CIR2++ model, the price of a zero-coupon bond with maturity T at time $t \in [0, T]$ is given by*

$$P(t,T) = \exp\left\{ -\int_t^T \varphi(u)\mathrm{d}u \right\} P^{\mathrm{CIR2}}(t,T),$$

where P^{CIR2} is P from Theorem 6.20.

THEOREM 6.28 (Forward rate in the CIR2++ model). *In the CIR2++ model, the instantaneous forward rate with maturity T is given by*

$$f(t,T) = \varphi(T) + f^{\mathrm{CIR2}}(t,T),$$

where f^{CIR2} is f from Theorem 6.23.

THEOREM 6.29 (Calibration in the CIR2++ model). *If the CIR2++ model is calibrated to a given interest rate structure* $\{f^{\mathrm{M}}(0,t) : t \geq 0\}$*, then*

$$\varphi(t) = f^{\mathrm{M}}(0,t) - f^{\mathrm{CIR2}}(0,t),$$

where f^{CIR2} *is* f *from Theorem 6.23.*