CHAPTER 7

Heath–Jarrow–Morton Framework

7.1. Heath–Jarrow–Morton Model

DEFINITION 7.1 (Forward-rate dynamics in the HJM model). In the *Heath–Jarrow–Morton model*, briefly *HJM model*, the instantaneous forward interest rate with maturity T is assumed to satisfy the stochastic differential equation

$$\mathrm{d}f(t,T) = \alpha(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}W(t),$$

where α and σ are adapted and W is a Brownian motion under the risk-neutral measure.

THEOREM 7.2 (Bond-price dynamics in the HJM model). *In the HJM model, the price of a zero-coupon bond with maturity T satisfies the stochastic differential equation*

$$\mathrm{d}P(t,T) = \left(r(t) + A(t,T) + \frac{1}{2}\Sigma^2(t,T)\right)P(t,T)\mathrm{d}t + \Sigma(t,T)P(t,T)\mathrm{d}W(t),$$

where

$$A(t,T) = -\int_t^T \alpha(t,u)\mathrm{d}u \quad \text{and} \quad \Sigma(t,T) = -\int_t^T \sigma(t,u)\mathrm{d}u.$$

THEOREM 7.3 (Bond-price dynamics implying HJM model). *If the price of a zero-coupon bond with maturity T satisfies the stochastic differential equation*

$$\mathrm{d}P(t,T) = m(t,T)P(t,T)\mathrm{d}t + v(t,T)P(t,T)\mathrm{d}W(t),$$

where m and v are adapted, then the forward-rate dynamics are as in the HJM model with

$$\alpha(t,T) = v(t,T)v_T(t,T) - m_T(t,T) \quad \text{and} \quad \sigma(t,T) = -v_T(t,T).$$

THEOREM 7.4 (Drift restriction in the HJM model). *In the HJM model, we necessarily have*

$$A(t,T) = -\frac{1}{2}\Sigma^2(t,T) \quad \text{and} \quad \alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)\mathrm{d}u.$$

THEOREM 7.5 (Bond-price dynamics in the HJM model). *In the HJM model, the price of a zero-coupon bond with maturity T satisfies the stochastic differential equations*

$$\mathrm{d}P(t,T) = r(t)P(t,T)\mathrm{d}t + \Sigma(t,T)P(t,T)\mathrm{d}W(t)$$

and

$$\mathrm{d}\frac{1}{P(t,T)} = \frac{\Sigma^2(t,T) - r(t)}{P(t,T)}\mathrm{d}t - \frac{\Sigma(t,T)}{P(t,T)}\mathrm{d}W(t).$$

THEOREM 7.6 (T-forward measure dynamics of the forward rate in the HJM model). *Under the T-forward measure $\mathbb{Q}^T$, the instantaneous forward interest rate with maturity T in the HJM model satisfies*

$$\mathrm{d}f(t,T) = \sigma(t,T)\mathrm{d}W^T(t),$$

where the $\mathbb{Q}^T$-Brownian motion W^T is defined by

$$\mathrm{d}W^T(t) = \mathrm{d}W(t) - \Sigma(t,T)\mathrm{d}t.$$

THEOREM 7.7 (Forward-rate dynamics in the HJM model). *In the HJM model, the simply-compounded forward interest rate for the period $[T,S]$ satisfies the stochastic differential equation*

$$\mathrm{d}F(t;T,S) = \left(F(t;T,S) + \frac{1}{\tau(T,S)}\right)\left(\Sigma(t,T) - \Sigma(t,S)\right)\mathrm{d}W^S(t).$$

THEOREM 7.8 (Zero-coupon bond in the HJM model). *Let $0 \le t \le T \le S$. In the HJM model, the price of a zero-coupon bond with maturity S at time T is given by*

$$P(T,S) = \frac{P(t,S)}{P(t,T)}e^Z,$$

where

$$\begin{aligned} Z &= -\frac{1}{2}\int_t^T \left(\Sigma^2(u,S) - \Sigma^2(u,T)\right)\mathrm{d}u + \int_t^T \left(\Sigma(u,S) - \Sigma(u,T)\right)\mathrm{d}W(u) \\ &= -\frac{1}{2}\int_t^T \left(\Sigma(u,S) - \Sigma(u,T)\right)^2 \mathrm{d}u + \int_t^T \left(\Sigma(u,S) - \Sigma(u,T)\right)\mathrm{d}W^T(u). \end{aligned}$$

7.2. Gaussian HJM Model

DEFINITION 7.9 (Gaussian HJM Model). A *Gaussian HJM model* is an HJM model in which σ is a deterministic function.

THEOREM 7.10 (Option on a zero-coupon bond in a Gaussian HJM model). *In a Gaussian HJM model, the price of a European call option with strike K and maturity T and written on a zero-coupon bond with maturity S at time $t \in [0,T]$ is given by*

$$\mathrm{ZBC}(t,T,S,K) = P(t,S)\Phi(h) - KP(t,T)\Phi(h-\sigma^*),$$

where

$$\sigma^* = \sqrt{\int_t^T \left(\Sigma(u,S) - \Sigma(u,T)\right)^2 \mathrm{d}u}$$

and

$$h = \frac{1}{\sigma^*}\ln\left(\frac{P(t,S)}{P(t,T)K}\right) + \frac{\sigma^*}{2}.$$

The price of a corresponding put option is given by

$$\mathrm{ZBP}(t,T,S,K) = KP(t,T)\Phi(-h+\sigma^*) - P(t,S)\Phi(-h).$$

DEFINITION 7.11 (Futures price). The *futures price* at time t of an asset whose value at time $T \geq t \geq 0$ is $X(T)$ is given by

$$\mathrm{Fut}(t,T) = \mathbb{E}(X(T)|\mathcal{F}(t)).$$

THEOREM 7.12 (Futures contract on a zero-coupon bond in a Gaussian HJM model). *In a Gaussian HJM model, the price of a futures contract with maturity T on a zero-coupon bond at time T with maturity S is given by*

$$\mathrm{FUT}(t,T,S) = \frac{P(t,S)}{P(t,T)}\exp\left\{\int_t^T \Sigma(u,T)\left(\Sigma(u,T) - \Sigma(u,S)\right)\mathrm{d}u\right\}.$$

7.3. Ritchken–Sankarasubramanian Model

DEFINITION 7.13 (HJM model with separable volatility). An HJM model with *separable volatility* is an HJM model in which there exist positive functions ξ and η such that

$$\sigma(t,T) = \xi(t)\eta(T).$$

Theorem 7.14 (Zero-coupon bond in an HJM model with separable volatility). *In an HJM model with separable volatility, the price of a zero-coupon bond with maturity T at time $t \in [0,T]$ is given by*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{ f(0,t)B(t,T) - \frac{1}{2}\phi(t)B^2(t,T) \right\} e^{-r(t)B(t,T)},$$

where

$$\phi(t) = \int_0^t \sigma^2(u,t)\mathrm{d}u \quad \text{and} \quad B(t,T) = \frac{1}{\eta(t)} \int_t^T \eta(u)\mathrm{d}u.$$

Theorem 7.15 (Short-rate dynamics in an HJM model with separable volatility). *In an HJM model with separable volatility, the short rate satisfies the stochastic differential equation*

$$\begin{aligned} \mathrm{d}r(t) &= \left\{ \frac{\partial f(0,t)}{\partial t} + \phi(t) \right\} \mathrm{d}t + \frac{r(t) - f(0,t)}{\eta(t)} \mathrm{d}\eta(t) \\ &\quad + \xi(t)(\mathrm{d}\eta(t))(\mathrm{d}W(t)) + \sigma(t,t)\mathrm{d}W(t), \end{aligned}$$

where ϕ is as in Theorem 7.14.

Corollary 7.16 (Short-rate dynamics in a Gaussian HJM model with separable volatility). *In an HJM model with separable volatility in which η is deterministic, the short rate satisfies the stochastic differential equation*

$$\mathrm{d}r(t) = \left\{ \frac{\partial f(0,t)}{\partial t} - f(0,t)\frac{\eta'(t)}{\eta(t)} + \phi(t) + r(t)\frac{\eta'(t)}{\eta(t)} \right\} \mathrm{d}t + \sigma(t,t)\mathrm{d}W(t),$$

where ϕ is as in Theorem 7.14.

Theorem 7.17 (Option on a zero-coupon bond in a Gaussian HJM model with separable volatility). *In a Gaussian HJM model with separable volatility, the price of a European call option with strike K and maturity T and written on a zero-coupon bond with maturity S at time $t \in [0,T]$ is given by*

$$\mathrm{ZBC}(t,T,S,K) = P(t,S)\Phi(h) - KP(t,T)\Phi(h - \sigma^*),$$

where

$$\sigma^* = B(T,S)\sqrt{\int_t^T \sigma^2(u,T)\mathrm{d}u} \quad \text{and} \quad h = \frac{1}{\sigma^*} \ln\left(\frac{P(t,S)}{P(t,T)K} \right) + \frac{\sigma^*}{2}$$

with B as in Theorem 7.14. The price of a corresponding put option is given by

$$\mathrm{ZBP}(t,T,S,K) = KP(t,T)\Phi(-h + \sigma^*) - P(t,S)\Phi(-h).$$

THEOREM 7.18 (Futures contract on a zero-coupon bond in a Gaussian HJM model with separable volatility). *In a Gaussian HJM model with separable volatility, the price of a futures contract with maturity T on a zero-coupon bond at time T with maturity S is given by*

$$\begin{aligned}\mathrm{FUT}(t,T,S) &= \frac{P(t,S)}{P(t,T)}\exp\left\{-B(T,S)\int_t^T \sigma(u,u)\sigma(u,T)B(u,T)\mathrm{d}u\right\}\\ &= \frac{P(t,S)}{P(t,T)}\exp\left\{-\left(\int_T^S \eta(u)\mathrm{d}u\right)\left(\int_t^T \eta(s)\int_t^s \xi^2(u)\mathrm{d}u\mathrm{d}s\right)\right\}.\end{aligned}$$

DEFINITION 7.19 (Ritchken–Sankarasubramanian model). The *Ritchken–Sankarasubramanian model* is an HJM model with separable volatility for which there exist functions σ and k such that

$$\xi(t) = \sigma(t)\exp\left\{\int_0^t k(u)\mathrm{d}u\right\} \quad \text{and} \quad \eta(t) = \exp\left\{-\int_0^t k(u)\mathrm{d}u\right\}.$$

THEOREM 7.20 (Zero-coupon bond in the Ritchken–Sankarasubramanian model). *In the Ritchken–Sankarasubramanian model, the price of a zero-coupon bond with maturity T at time $t \in [0,T]$ is given by*

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left\{f(0,t)B(t,T) - \frac{1}{2}\phi(t)B^2(t,T)\right\}e^{-r(t)B(t,T)},$$

where

$$\phi(t) = \int_0^t \sigma^2(u)\exp\left\{-2\int_u^t k(v)\mathrm{d}v\right\}\mathrm{d}u$$

and

$$B(t,T) = \int_t^T \exp\left\{-\int_t^s k(u)\mathrm{d}u\right\}\mathrm{d}s.$$

THEOREM 7.21 (Short-rate dynamics in the Ritchken–Sankarasubramanian model). *In a Ritchken–Sankarasubramanian model in which k is deterministic and positive, the short rate satisfies the stochastic differential equation*

$$\mathrm{d}r(t) = \left(k(t)f(0,t) + \frac{\partial f(0,t)}{\partial t} + \phi(t) - k(t)r(t)\right)\mathrm{d}t + \sigma(t)\mathrm{d}W(t)$$

with ϕ as in Theorem 7.20.

DEFINITION 7.22 (Gaussian HJM model with exponentially damped volatility). A Gaussian HJM model with *exponentially damped volatility* is a Ritchken–Sankarasubramanian model in which the functions σ and k are positive constants.

THEOREM 7.23 (The Gaussian HJM model with exponentially damped volatility and the Hull–White model). *Suppose r is the short rate in a Gaussian HJM model with exponentially damped volatility. Then r is equal to the short rate in the corresponding calibrated Hull–White model.*

REMARK 7.24. Since for a Gaussian HJM model with exponentially damped volatility we have

$$\sigma(t,T) = \sigma e^{-k(T-t)}, \quad B(t,T) = \frac{1-e^{-k(T-t)}}{k},$$
$$\int_t^T \sigma^2(u,T)\mathrm{d}u = \frac{\sigma^2}{2k}\left(1-e^{-2k(T-t)}\right), \quad \phi(t) = \frac{\sigma^2}{2k}\left(1-e^{-2kt}\right),$$

we may use Theorem 7.23 to show that

- Theorem 7.20 implies Theorem 5.12;
- Theorem 7.17 implies Theorem 5.13;
- Theorem 7.18 implies for the Hull–White model
$$\mathrm{FUT}(t,T,S) = \frac{P(t,S)}{P(t,T)} \exp\left(-\frac{\sigma^2}{2} B(T,S)B^2(t,T)\right).$$

DEFINITION 7.25 (Gaussian HJM model with constant volatility). A Gaussian HJM model with *constant volatility* is a Ritchken–Sankarasubramanian model in which σ is a positive constant and $k=0$.

THEOREM 7.26 (The Gaussian HJM model with constant volatility and the Ho–Le model). *Suppose r is the short rate in a Gaussian HJM model with constant volatility. Then r is equal to the short rate in the corresponding calibrated Ho–Le model.*

REMARK 7.27. Since for a Gaussian HJM model with constant volatility we have

$$\sigma(t,T) = \sigma, \quad B(t,T) = T-t, \quad \int_t^T \sigma^2(u,T)\mathrm{d}u = \sigma^2(T-t), \quad \phi(t) = \sigma^2 t,$$

we may use Theorem 7.26 to show that

- Theorem 7.20 implies Theorem 5.4;
- Theorem 7.17 implies the formula for ZBC from Theorem 5.2;
- Theorem 7.18 implies for the Ho–Le model
$$\mathrm{FUT}(t,T,S) = \frac{P(t,S)}{P(t,T)} \exp\left(-\frac{\sigma^2}{2}(S-T)(T-t)^2\right).$$

7.4. Mercurio–Moraleda Model

DEFINITION 7.28 (Gaussian HJM model with volatility depending on time to maturity). A Gaussian HJM model with *volatility depending on time to maturity* is an HJM model in which there exists a deterministic function h such that

$$\sigma(t,T) = h(T-t).$$

THEOREM 7.29 (Option on a zero-coupon bond in a Gaussian HJM model with volatility depending on time to maturity). *In a Gaussian HJM model with volatility depending on time to maturity, the price of a European call option with strike K and maturity T and written on a zero-coupon bond with maturity S at time $t \in [0,T]$ is given by*

$$\mathrm{ZBC}(t,T,S,K) = P(t,S)\Phi(h) - KP(t,T)\Phi(h-\sigma^*),$$

where

$$\sigma^* = \sqrt{\int_0^\tau \left(\int_u^{u+\mu} h(x)\mathrm{d}x\right)^2 \mathrm{d}u} \quad \textit{with} \quad \tau = T-t \quad \textit{and} \quad \mu = S-T$$

and

$$h = \frac{1}{\sigma^*}\ln\left(\frac{P(t,S)}{P(t,T)K}\right) + \frac{\sigma^*}{2}.$$

The price of a corresponding put option is given by

$$\mathrm{ZBP}(t,T,S,K) = KP(t,T)\Phi(-h+\sigma^*) - P(t,S)\Phi(-h).$$

THEOREM 7.30 (Futures contract on a zero-coupon bond in a Gaussian HJM model with volatility depending on time to maturity). *In a Gaussian HJM model with volatility depending on time to maturity, the price of a futures contract with maturity T on a zero-coupon bond at time T with maturity S is given by*

$$\mathrm{FUT}(t,T,S) = \frac{P(t,S)}{P(t,T)}\exp\left\{\int_0^\tau \left(\int_0^u h(x)\mathrm{d}x\right)\left(\int_u^{u+\mu} h(x)\mathrm{d}x\right)\mathrm{d}u\right\}$$

with τ and μ as in Theorem 7.29.

DEFINITION 7.31 (Mercurio–Moraleda model). The *Mercurio–Moraleda model* is a Gaussian HJM model with volatility depending on time to maturity for which there exist constants $\sigma, \gamma, \lambda > 0$ such that

$$h(x) = \sigma(1+\gamma x)e^{-\frac{\lambda}{2}x}.$$

THEOREM 7.32 (Option on a zero-coupon bond in the Mercurio–Moraleda model). *In the Mercurio–Moraleda model, the price of a European call option with strike K and maturity T and written on a zero-coupon bond with maturity S at time $t \in [0, T]$ is given by*

$$\mathrm{ZBC}(t, T, S, K) = P(t, S)\Phi(h) - KP(t, T)\Phi(h - \sigma^*),$$

where

$$\sigma^* = \frac{2\sigma}{\lambda^{7/2}} \sqrt{(\alpha^2\lambda^2 + 2\alpha\beta\lambda + 2\beta^2)(1 - e^{-\lambda\tau}) - \lambda\beta\tau(2\alpha\lambda + 2\beta + \beta\lambda\tau)e^{-\lambda\tau}}$$

and

$$h = \frac{1}{\sigma^*} \ln\left(\frac{P(t, S)}{P(t, T)K}\right) + \frac{\sigma^*}{2}$$

with

$$\alpha = (\lambda + 2\gamma)(1 - e^{-\frac{\lambda}{2}\mu}) - \gamma\lambda\mu e^{-\frac{\lambda}{2}\mu}, \quad \beta = \gamma\lambda(1 - e^{-\frac{\lambda}{2}\mu})$$

and τ and μ are as in Theorem 7.29. The price of a corresponding put option is given by

$$\mathrm{ZBP}(t, T, S, K) = KP(t, T)\Phi(-h + \sigma^*) - P(t, S)\Phi(-h).$$

THEOREM 7.33 (Futures contract on a zero-coupon bond in the Mercurio–Moraleda model). *In the Mercurio–Moraleda model, the price of a futures contract with maturity T on a zero-coupon bond at time T with maturity S is given by*

$$\mathrm{FUT}(t, T, S) = \frac{P(t, S)}{P(t, T)} \exp\left(\frac{4\sigma^2}{\lambda^4} z\right)$$

with

$$\begin{aligned} z \quad = \quad & \frac{\alpha\alpha_0\lambda^2 + \alpha_0\beta\lambda + \alpha\beta_0\lambda + 2\beta\beta_0}{\lambda^3}(e^{-\lambda\tau} - 1) + \frac{\alpha_0\beta\lambda + \beta_0\alpha\lambda + 2\beta\beta_0}{\lambda^2}\tau e^{-\lambda\tau} \\ & + \frac{\beta\beta_0}{\lambda}\tau^2 e^{-\lambda\tau} + \frac{2\alpha_0(\alpha\lambda + 2\beta)}{\lambda^2}\left(1 - e^{-\frac{\lambda}{2}\tau}\right) - \frac{2\beta\alpha_0}{\lambda}\tau e^{-\frac{\lambda}{2}\tau}, \end{aligned}$$

where α, β, τ, μ are as in Theorem 7.32 and

$$\alpha_0 = \lambda + 2\gamma \quad \textit{and} \quad \beta_0 = \gamma\lambda.$$