CHAPTER 8

Market Models

8.1. Lognormal Forward-LIBOR Model

Remark 8.1. Recall from Example 3.10 that $F(t;T,S)$ is a martingale under $\mathbb{Q}^S$.

Definition 8.2 (Forward-rate dynamics in the LFM model). In the *lognormal forward-LIBOR model*, briefly *LFM model*, the simply-compounded forward interest rate for the period $[T,S]$ is assumed to satisfy the stochastic differential equation

$$\mathrm{d}F(t;T,S) = \sigma(t;T,S)F(t;T,S)\mathrm{d}W^S(t),$$

where σ is deterministic and W^S is a Brownian motion under the S-forward measure.

Theorem 8.3 (Pricing of caplets in the LFM model). *In the LFM model, the price of a caplet with nominal value N, cap rate K, expiry time T, and maturity time S is given by*

$$\mathrm{Cpl}(t,T,S,N,K) = NP(t,S)\tau(T,S)\,\mathrm{Bl}\left(K, F(t;T,S), \sqrt{\int_t^T \sigma^2(u;T,S)\mathrm{d}u}\right),$$

where

$$\mathrm{Bl}(K,F,v) = F\Phi\left(\frac{\ln\left(\frac{F}{K}\right)+\frac{v^2}{2}}{v}\right) - K\Phi\left(\frac{\ln\left(\frac{F}{K}\right)-\frac{v^2}{2}}{v}\right).$$

Theorem 8.4 (Pricing of caps in the LFM model). *In the LFM model, the price of a cap with nominal value N, cap rate K, and the set of times $\mathcal{T}$, is given by*

$$\mathrm{Cap}(t,\mathcal{T},N,K) = N\sum_{i=\alpha+1}^{\beta} P(t,T_i)\tau_i\,\mathrm{Bl}\left(K, F_i(t), \sqrt{\int_t^{T_{i-1}} \sigma_i^2(u)\mathrm{d}u}\right),$$

where for $\alpha+1 \le i \le \beta$

$$F_i(t) = F(t;T_{i-1},T_i), \quad \tau_i = \tau(T_{i-1},T_i), \quad \sigma_i(t) = \sigma(t;T_{i-1},T_i).$$

THEOREM 8.5 (Brownian motions in the LFM model under different forward measures). *Let $S > T$. Let W^T and W^S be a $\mathbb{Q}^T$-Brownian motion and a $\mathbb{Q}^S$-Brownian motion, respectively. In the LFM model, we then have*

$$\mathrm{d}W^S(t) - \mathrm{d}W^T(t) = \frac{\tau(T,S)F(t;T,S)\sigma(t;T,S)}{1+\tau(T,S)F(t;T,S)}\mathrm{d}t.$$

THEOREM 8.6 (Forward-rate dynamics in the LFM model). *Let $i, k \in \{\alpha, \ldots, \beta\}$. In the* LFM model, *$F_k$ satisfies the following stochastic differential equations:*

(i) *If $k = i$, then*

$$\mathrm{d}F_k(t) = \sigma_k(t)F_k(t)\mathrm{d}W^{T_i}(t).$$

(ii) *If $k > i$, then*

$$\mathrm{d}F_k(t) = \sigma_k(t)F_k(t)\mathrm{d}W^{T_i}(t) + \left(\sum_{j=i+1}^{k} \frac{\tau_j\sigma_j(t)\sigma_k(t)F_j(t)F_k(t)}{1+\tau_jF_j(t)}\right)\mathrm{d}t.$$

(iii) *If $k < i$, then*

$$\mathrm{d}F_k(t) = \sigma_k(t)F_k(t)\mathrm{d}W^{T_i}(t) - \left(\sum_{j=k+1}^{i} \frac{\tau_j\sigma_j(t)\sigma_k(t)F_j(t)F_k(t)}{1+\tau_jF_j(t)}\right)\mathrm{d}t.$$

8.2. Lognormal Forward-Swap Model

REMARK 8.7. Recall from Example 3.11 that $S_{\alpha,\beta}(t)$ is a martingale under $\mathbb{Q}^{\alpha,\beta}$.

DEFINITION 8.8 (Forward swap rate dynamics in the LSM model). In the *lognormal forward-swap model*, briefly *LSM model*, the forward swap rate for $\mathcal{T} = \{T_\alpha, \ldots, T_\beta\}$ is assumed to satisfy the stochastic differential equation

$$\mathrm{d}S_{\alpha,\beta}(t) = \sigma_{\alpha,\beta}(t)S_{\alpha,\beta}(t)\mathrm{d}W^{\alpha,\beta}(t),$$

where σ is deterministic and $W^{\alpha,\beta}$ is a Brownian motion under $\mathbb{Q}^{\alpha,\beta}$.

THEOREM 8.9 (Pricing of swaptions in the LSM model). *In the LSM model, the price of a European payer swaption with swaption maturity $T = T_\alpha$ on an IRS depending on the nominal value N, the fixed rate K, and the set of times $\mathcal{T}$ is given by*

$$\mathrm{PS}(t,T,\mathcal{T},N,K) = N\left(\sum_{i=\alpha+1}^{\beta}\tau_iP(t,T_i)\right)\mathrm{Bl}\left(K, S_{\alpha,\beta}(t), \sqrt{\int_t^T \sigma^2_{\alpha,\beta}(u)\mathrm{d}u}\right).$$