CHAPTER 9

The Volatility Smile

9.1. The Smile Problem

Remark 9.1. Assuming the LFM, we have

$$\mathrm{Cpl}(0, T, S, N, K_1) = NP(0, S)\tau(T, S)\,\mathrm{Bl}(K_1, F(0; T, S), v_S(T))$$

and

$$\mathrm{Cpl}(0, T, S, N, K_2) = NP(0, S)\tau(T, S)\,\mathrm{Bl}(K_2, F(0; T, S), v_S(T)),$$

where the volatility parameter is given by

$$v_S(T) = \sqrt{\int_0^T \sigma^2(u; T, S)\mathrm{d}u}.$$

However, for $K_1 \neq K_2$, this is not realistic to hold with the same volatility parameter.

Definition 9.2 (The Volatility Smile). If caplet prices are given by

$$\mathrm{Cpl}(0, T, S, N, K) = NP(0, S)\tau(T, S)\,\mathrm{Bl}(K, F(0; T, S), v_S(T, K)),$$

then the curve

$$K \mapsto \frac{v_S(T, K)}{\sqrt{T}}$$

is called the *volatility smile* of the T-expiry caplet.

Remark 9.3. In the LFM, the volatility smile is “flat”. However, the volatility smile is commonly seen to exhibit “smiley” or “skewed” shapes.

9.2. Shifted Lognormal Model

Definition 9.4 (Forward-rate dynamics in the shifted lognormal model). In the *shifted lognormal model*, the simply-compounded forward interest rate for the period $[T, S]$ is assumed to satisfy the stochastic differential equation

$$\mathrm{d}F(t; T, S) = \sigma(t; T, S)\left(F(t; T, S) - \alpha\right)\mathrm{d}W^S(t),$$

where σ is deterministic, $\alpha \neq 0$, and W^S is a Brownian motion under the S-forward measure.

THEOREM 9.5 (Pricing of caplets in the shifted lognormal model). *In the shifted lognormal model, the price of a caplet with notional value N, cap rate K, expiry time T, and maturity time S is given by*

$$\begin{aligned}&\mathrm{Cpl}(t,T,S,N,K)\\&\qquad = NP(t,S)\tau(T,S)\,\mathrm{Bl}\left(K-\alpha, F(t;T,S)-\alpha, \sqrt{\int_t^T \sigma^2(u;T,S)\mathrm{d}u}\right).\end{aligned}$$

THEOREM 9.6 (The volatility smile in the shifted lognormal model). *If $\alpha > 0$, then the volatility smile in the shifted lognormal model is increasing. If $\alpha < 0$, then the volatility smile in the shifted lognormal model is decreasing.*

9.3. Brigo–Mercurio Local Volatility Model

DEFINITION 9.7 (Forward-rate dynamics in the Brigo–Mercurio local volatility model). In the *Brigo–Mercurio local volatility model*, the simply-compounded forward interest rate for the period $[T,S]$ is assumed to satisfy the stochastic differential equation

$$\mathrm{d}F(t;T,S) = \sigma(t, F(t;T,S))F(t;T,S)\mathrm{d}W^S(t),$$

where W^S is a Brownian motion under the S-forward measure $\mathbb{Q}^S$,

$$\sigma(t,y) = \sqrt{\frac{\sum_{i=1}^n \lambda_i v_i^2(t,y)p_t^i(y)}{\sum_{i=1}^n \lambda_i y^2 p_t^i(y)}}, \quad \sum_{i=1}^n \lambda_i = 1,$$

and

$$\lambda_i > 0, \quad p_t^i(y) = \frac{\mathrm{d}(\mathbb{Q}^S(G_i(t) \leq y))}{\mathrm{d}y}, \quad 1 \leq i \leq n$$

such that

$$\mathrm{d}G_i(t) = v_i(t, G_i(t))\mathrm{d}W^S(t), \quad G_i(0) = F(0;T,S), \quad 1 \leq i \leq n.$$

REMARK 9.8 (Fokker–Planck equation). Using the result that the pdf f_t of the solution of the stochastic differential equation

$$\mathrm{d}X(t) = \mu(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W(t)$$

satisfies the *Fokker–Planck equation*

$$\frac{\partial}{\partial t} f_t(y) = -\frac{\partial}{\partial y}(\mu(t,y)f_t(y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(t,y)f_t(y)),$$

we may show that the pdf p_t of $F(t;T,S)$ in the Brigo–Mercurio local volatility model is given by

$$p_t(y) = \sum_{i=1}^{n} \lambda_i p_t^i(y).$$

THEOREM 9.9 (Pricing of caplets in the Brigo–Mercurio local volatility model). *In the Brigo–Mercurio local volatility model, the price of a caplet with notional value N, cap rate K, expiry time T, and maturity time S is given by*

$$\mathrm{Cpl}(0,T,S,N,K) = NP(0,S)\tau(T,S)\sum_{i=1}^{n} \lambda_i \mathbb{E}^S((G_i(T) - K)^+).$$

9.4. Lognormal Mixture Model

DEFINITION 9.10 (LM model). A *lognormal mixture model*, briefly *LM model*, is a Brigo–Mercurio local volatility model in which

$$v_i(t,y) = \sigma_i(t)y, \quad 1 \le i \le n,$$

where σ_i are deterministic for all $1 \le i \le n$.

REMARK 9.11 (Probability density functions in the LM model). In the LM model, the probability density functions p_t^i are given by

$$p_t^i(y) = \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left(\ln\left(\frac{y}{F(0;T,S)}\right) + \frac{1}{2}V_i^2(t)\right)^2\right\},$$

where

$$V_i(t) = \sqrt{\int_0^t \sigma_i^2(u)\mathrm{d}u}, \quad 1 \le i \le n.$$

The forward-rate dynamics in the LM model is then given by

$$\mathrm{d}F(t;T,S) = \sigma(t, F(t;T,S))F(t;T,S)\mathrm{d}W^S(t),$$

where W^S is a Brownian motion under the S-forward measure $\mathbb{Q}^S$ and

$$\sigma(t,y)=\sqrt{\frac{\sum_{i=1}^{n}\lambda_i\sigma_i^2(t)p_t^i(y)}{\sum_{i=1}^{n}\lambda_i p_t^i(y)}}.$$

Note also that

$$\sigma^2(t,y)=\sum_{i=1}^{n}\Lambda_i(t,y)\sigma_i^2(t)\quad\text{with}\quad\Lambda_i>0\quad\text{such that}\quad\sum_{i=1}^{n}\Lambda_i=1.$$

THEOREM 9.12 (Pricing of caplets in the LM model). *In the LM model, the price of a caplet with notional value N, cap rate K, expiry time T, and maturity time S is given by*

$$\mathrm{Cpl}(0,T,S,N,K)=NP(0,S)\tau(T,S)\sum_{i=1}^{n}\lambda_i\,\mathrm{Bl}(K,F(0;T,S),V_i(T)),$$

where V_i are as in Remark 9.11.

9.5. Lognormal Mixture Model with Different Means

DEFINITION 9.13 (Forward-rate dynamics in the LMDM model). In the *logormal mixture model with different means*, briefly *LMDM model*, the simply-compounded forward interest rate for the period $[T,S]$ is assumed to satisfy the stochastic differential equation

$$\mathrm{d}F(t;T,S)=\sigma(t,F(t;T,S))F(t;T,S)\mathrm{d}W^S(t),$$

where W^S is a Brownian motion under the S-forward measure $\mathbb{Q}^S$,

$$\sigma(t,y)=\sqrt{\frac{\sum_{i=1}^{n}\lambda_i\sigma_i^2(t)p_t^i(y)}{\sum_{i=1}^{n}\lambda_i p_t^i(y)}+\frac{2\sum_{i=1}^{n}\lambda_i\mu_i(t)\int_y^\infty xp_t^i(x)\mathrm{d}x}{\sum_{i=1}^{n}\lambda_i y^2 p_t^i(y)}},\quad\sum_{i=1}^{n}\lambda_i=1,$$

μ_i and σ_i are deterministic for all $1\le i\le n$ such that $\sigma(t,y)$ is well defined and such that

$$\sum_{i=1}^{n}\lambda_i\exp\left\{\int_0^t\mu_i(u)\mathrm{d}u\right\}=1,$$

and

$$\lambda_i>0,\quad p_t^i(y)=\frac{\mathrm{d}(\mathbb{Q}^S(G_i(t)\le y))}{\mathrm{d}y},\quad 1\le i\le n$$

such that

$$\mathrm{d}G_i(t) = \mu_i(t)G_i(t)\mathrm{d}t + \sigma_i(t)G_i(t)\mathrm{d}W^S(t), \quad G_i(0) = F(0;T,S), \quad 1 \le i \le n.$$

REMARK 9.14 (Probability density functions in the LMDM model). In the LMDM model, the probability density functions p_t^i are given by

$$p_t^i(y) = \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left(\ln\left(\frac{y}{F(0;T,S)}\right) - M_i(t) + \frac{1}{2}V_i^2(t)\right)^2\right\},$$

where

$$M_i(t) = \int_0^t \mu_i(u)\mathrm{d}u, \quad V_i(t) = \sqrt{\int_0^t \sigma_i^2(u)\mathrm{d}u}, \quad 1 \le i \le n.$$

Also, the pdf p_t of $F(t;T,S)$ in the LMDM model is given by

$$p_t(y) = \sum_{i=1}^n \lambda_i p_t^i(y).$$

THEOREM 9.15 (Pricing of caplets in the LMDM model). *In the LMDM model, the price of a caplet with notional value N, cap rate K, expiry time T, and maturity time S is given by*

$$\mathrm{Cpl}(0,T,S,N,K) = NP(0,S)\tau(T,S)\sum_{i=1}^n \lambda_i e^{M_i(T)}\,\mathrm{Bl}(Ke^{-M_i(T)}, F(0;T,S), V_i(T)),$$

where M_i and V_i are as in Remark 9.14.

9.6. Second Brigo–Mercurio Local Volatility Model

DEFINITION 9.16 (Second Brigo–Mercurio local volatility model). In the *second Brigo–Mercurio local volatility model*, the simply-compounded forward interest rate for the period $[T,S]$ is assumed to be given in the form

$$F(t) = F(t;T,S) = h(t, W^S(t)),$$

where W^S is a Brownian motion under the S-forward measure and h is a positive function in two variables which is continuously differentiable in the first variable and twice continuously differentiable in the second variable, strictly increasing in the second variable satisfying

$$\lim_{x\to\infty} h^{-1}(T,x) = \infty,$$

where we write $h^{-1}(t,x) = y$ if $h(t,y) = x$, and such that F is a martingale under W^S.

THEOREM 9.17 (Forward-rate dynamics in the second Brigo–Mercurio local volatility model). *In the second Brigo–Mercurio local volatility model, the simply-compounded forward interest rate for the period $[T, S]$ satisfies the stochastic differential equation*

$$\mathrm{d}F(t) = \frac{\partial}{\partial w} h(t, h^{-1}(t, F(t)))\mathrm{d}W^S(t),$$

where W^S is a Brownian motion under the S-forward measure.

LEMMA 9.18 (Transition density in the second Brigo–Mercurio local volatility model). *In the second Brigo–Mercurio local volatility model, we have*

$$\mathbb{Q}^S(F(T) \leq x | F(t) = y) = \Phi\left(\frac{h^{-1}(T, x) - h^{-1}(t, y)}{\sqrt{T-t}}\right),$$

and the density of $F(T)$ conditional on $F(t) = y$ is given by

$$p(t, y; T, x) = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(h^{-1}(T, x) - h^{-1}(t, y))^2}{2(T-t)}\right) \frac{\mathrm{d}}{\mathrm{d}x} h^{-1}(T, x).$$

THEOREM 9.19 (Pricing of caplets in the second Brigo–Mercurio local volatility model). *In the second Brigo–Mercurio local volatility model, the price of a caplet with notional value N, cap rate K, expiry time T, and maturity time S is given by*

$$\begin{aligned}
&\mathrm{Cpl}(t, T, S, N, K) \\
&= NP(t, S)\tau(T, S) \left\{ \frac{\int_{h^{-1}(T,K)-h^{-1}(t,F(t))}^{\infty} h(T, h^{-1}(t, F(t)) + w) e^{-\frac{w^2}{2(T-t)}} \mathrm{d}w}{\sqrt{2\pi(T-t)}} \right. \\
&\qquad \left. - K\Phi\left(\frac{h^{-1}(t, F(t)) - h^{-1}(T, K)}{\sqrt{T-t}}\right) \right\}.
\end{aligned}$$

The price of the caplet at time 0 is given by

$$\begin{aligned}
&\mathrm{Cpl}(0, T, S, N, K) \\
&= NP(0, S)\tau(T, S) \left\{ \frac{1}{\sqrt{2\pi T}} \int_{h^{-1}(T,K)}^{\infty} h(T, w) e^{-\frac{w^2}{2T}} \mathrm{d}w - K\Phi\left(-\frac{h^{-1}(T, K)}{\sqrt{T}}\right) \right\}.
\end{aligned}$$

THEOREM 9.20 (Volatility smile in the second Brigo–Mercurio local volatility model). *The volatility smile in the second Brigo–Mercurio local volatility model satisfies the equation*

$$\frac{\mathrm{d}}{\mathrm{d}K}\left(\mathrm{Bl}(K, F(0), v(K))\right) = -\Phi\left(-\frac{h^{-1}(T, K)}{\sqrt{T}}\right).$$

9.7. Geometric Brownian Motion Mixture Model

Definition 9.21 (GBM mixture model). The *geometric Brownian motion mixture model*, briefly *GBM mixture model*, is a second Brigo–Mercurio local volatility model in which the function h is given by

$$h(t,w) = \sum_{i=1}^{n} \alpha_i e^{-\frac{1}{2}\beta_i^2 t + \beta_i w},$$

where $\alpha_i, \beta_i > 0$ for all $1 \le i \le n$.

Remark 9.22 (GBM mixture model). Indeed the given function h satisfies all necessary requirements. Moreover, F satisfies

$$\mathrm{d}F(t) = \sigma(t, F(t))F(t)\mathrm{d}W^S(t),$$

where

$$\sigma(t,y) = \frac{\sum_{i=1}^{n} \alpha_i \beta_i e^{-\frac{1}{2}\beta_i^2 t + \beta_i h^{-1}(t,y)}}{\sum_{i=1}^{n} \alpha_i e^{-\frac{1}{2}\beta_i^2 t + \beta_i h^{-1}(t,y)}}.$$

Note also that

$$\sigma(t,y) = \sum_{i=1}^{n} \Lambda_i(t,y)\beta_i \quad \text{with} \quad \Lambda_i > 0 \quad \text{such that} \quad \sum_{i=1}^{n} \Lambda_i = 1$$

so that the local volatility may be viewed as a stochastic average of the basic volatilities β_i.

Theorem 9.23 (Pricing of caplets in the GBM mixture model). *In the GBM mixture model, the price of a caplet with notional value N, cap rate K, expiry time T, and maturity time S is given by*

$$\begin{aligned} \mathrm{Cpl}(t,T,S,N,K) &= NP(t,S)\tau(T,S)\times \\ &\times \left\{ \sum_{i=1}^{n} \alpha_i e^{-\frac{1}{2}\beta_i^2 t + \beta_i h^{-1}(t,F(t))} \Phi\left(\frac{\beta_i (T-t) - h^{-1}(T,K) + h^{-1}(t,F(t))}{\sqrt{T-t}} \right) \right. \\ &\qquad \left. - K\Phi\left(\frac{h^{-1}(t,F(t)) - h^{-1}(T,K)}{\sqrt{T-t}} \right) \right\}. \end{aligned}$$

The price of the caplet at time 0 *is given by*

$$\begin{aligned}&\mathrm{Cpl}(0,T,S,N,K)\\&\quad= NP(0,S)\tau(T,S)\left\{\sum_{i=1}^{n}\alpha_i\Phi\left(\frac{\beta_i T - h^{-1}(T,K)}{\sqrt{T}}\right) - K\Phi\left(-\frac{h^{-1}(T,K)}{\sqrt{T}}\right)\right\}.\end{aligned}$$