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Winter Term 10/11

Institute of
Mathematical Finance

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Fixed Income Models

Exercise Sheet 2

(Due: Tuesday 11/16/2010)

6. **(Ito's formula)**
Let W be a Brownian motion. For arbitrary $n \in \mathbb{N}$ find a formula for
$$\int_0^t W^n(s)\mathrm{d}W(s).$$

7. **(Ito's formula, Integration by parts)**
Let W be a Brownian motion. Prove the following "integration by parts" formula for $n \in \mathbb{N}$:
$$\int_0^t s^n \mathrm{d}W(s) = s^n W(t) - \int_0^t n s^{n-1} W(s)\mathrm{d}s.$$

8. **(Brownian motion)**
Show that if W is a Brownian motion, then
$$\mathbb{E}\left(W^2(t)\right) = t.$$

9. **(Brownian motion, Covariance)**
Show that for a Brownian motion W, the covariance between two time points s and t can be computed as
$$\mathbb{C}\mathrm{ov}\left(W(s), W(t)\right) = \min\left\{s, t\right\}.$$

10. **(Brownian motion)**
Let Y be a standard normally distributed random variable and define $X(t) = \sqrt{t}Y$. Is the process X a Brownian motion?

11. **(Log-normal distribution)**
Let Y be a normally distributed random variable with $\mathbb{E}\left(Y\right) = \mu$ and $\mathbb{V}\left(Y\right) = \sigma^2$. Show that
$$\mathbb{E}\left(e^Y\right) = e^{\mu + \frac{\sigma^2}{2}}.$$

12. **(Generalized geometric Brownian motion)**
Compute
$$\mathbb{E}\left(X(t)|\mathcal{F}(s)\right)$$
and
$$\mathbb{V}\left(X(t)|\mathcal{F}(s)\right),$$
where X is the solution of the SDE
$$\mathrm{d}X(t) = \mu(t)X(t)\mathrm{d}t + \sigma(t)X(t)\mathrm{d}W(t).$$

13. **(Stochastic Leibniz rule)**
Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Consider the two processes Y and Z with
$$\mathrm{d}Y(t) = Y(t)\left(\mu \mathrm{d}t + \sigma \mathrm{d}W(t)\right)$$
and
$$Z(t) = \exp\left(-\frac{1}{2}\frac{\mu^2}{\sigma^2}t - \frac{\mu}{\sigma}W(t)\right).$$
Determine the SDE that is satisfied by YZ.

14. **(Option price)**
Let $K > 0$. Assume Y is a lognormally distributed random variable with $\mathbb{E}\left(\ln(Y)\right) = M$ and $\mathbb{V}\left(\ln(Y)\right) = V^2$. Show that
$$\mathbb{E}\left((Y-K)^+\right) = e^{M + \frac{V^2}{2}}\Phi\left(\frac{M - \ln(K) - V^2}{V}\right) - K\Phi\left(\frac{M - \ln(K)}{V}\right),$$
where Φ is the cdf of the standard normal distribution.