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Winter Term 10/11

Institute of Mathematical Finance

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Fixed Income Models

Exercise Sheet 4

(Due: Tuesday 11/30/2010)

24. **(CIR, Itô's formula)**
Let $W_1, \ldots, W_d$ be independent Brownian motions and let k and σ be positive constants. For $j = 1, \ldots, d$, let X_j be the solution of the Ornstein–Uhlenbeck SDE

$$\mathrm{d}X_j(t) = -\frac{k}{2}X_j(t)\mathrm{d}t + \frac{\sigma}{2}\mathrm{d}W_j(t).$$

Show that

$$X_j(t) = e^{-\frac{k}{2}t}\left[X_j(0) + \frac{\sigma}{2}\int_0^t e^{\frac{k}{2}u}\mathrm{d}W_j(u)\right]$$

and compute the expected value and the variance of $X_j(t)$.

25. (**CIR, Itô's formula**)
Let X_j be defined as in Exercise 24. Define

$$R(t) = \sum_{j=1}^{d} X_j^2(t)$$

and show that

$$\mathrm{d}R(t) = k\left(\theta - R(t)\right)\mathrm{d}t + \sigma\sqrt{R(t)}\mathrm{d}W(t),$$

where $\theta = \frac{d\sigma^2}{4k}$ and

$$W(t) = \sum_{j=1}^{d}\int_0^t \frac{X_j(s)}{\sqrt{R(s)}}\mathrm{d}W_j(s)$$

is a Brownian motion.

26. (**CIR, moment generating function**)
Let $R(t)$ be defined as in Exercise 25. Set

$$X_j(0) = \sqrt{\frac{R(0)}{d}}$$

for $R(0) > 0$. Show that the moment-generating function of $R(t)$ is given by

$$\mathbb{E}e^{uR(t)} = \left(\frac{1}{1-2v(t)u}\right)^{2k\theta/\sigma^2}\exp\left(\frac{e^{-kt}uR(0)}{1-2v(t)u}\right)$$

for all $u < \frac{1}{2v(t)}$, where $v(t) = \frac{\sigma^2}{4k}\left(1 - e^{-kt}\right)$.

27. (**CIR, positive values**)
Consider the short-rate model

$$\mathrm{d}r(t) = k\left(\theta - r(t)\right)\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W(t),$$

where $k, \theta, \sigma > 0$. Argue why the condition $2k\theta > \sigma^2$ is needed to ensure positive short rates.