



Fixed Income Models

Exercise Sheet 8
(Due: Tuesday 01./25/2011)

44. (Incompatibility between LSM and LFM)

For a given tenor structure $0 = T_0 < T_1 < T_2 < T_3$. Consider an LFM where the two LIBOR rates $L_1(t) := L(t, T_1)$ and $L_2(t) := L(t, T_2)$ follow lognormal martingales under their respective forward LIBOR measures, i.e.,

$$dL_i(t) = \sigma_i(t)L_i(t)dW^{T_{i+1}}, \quad i = 1, 2,$$

where the σ_i are deterministic functions and $W^{T_{i+1}}$ denotes a one-dimensional $\mathbb{Q}^{T_{i+1}}$ -Brownian motion. Derive the dynamics of the swap rate $S_{1,2}(t)$ under the forward swap measure \mathbb{Q}^{L_3} .

Hint: You do not need to consider any drift terms in your calculations. Why? From the result we can see that if LIBOR rates are lognormal under their respective forward measure, swap rates cannot be lognormal under their respective swap measures.

45. (Risk-neutral dynamics, LFM)

For a \mathbb{Q}^{T_k} -Brownian motion W^j and a \mathbb{Q}^{T_j} -Brownian motion W^i , $\rho_{k,j}$ is defined such that $dW^i(t)dW^j(t) = \rho_{k,j}dt$. Let $\beta(t)$ denote the index of the first forward rate that has not expired by t and $\sigma_j(t, u)$ denotes the instantaneous volatility of the instantaneous forward rate $f(t, u)$. Show that the risk-neutral dynamics of forward LIBOR rates in the LFM is

$$dF_k(t) = \mu_k(t)F_k(t)dt + \sigma_k(t)F_k(t)dW^{\beta(t)},$$

where W^0 denotes the Brownian-motion under the risk-neutral measure and

$$\mu_k(t) = \sum_{j=\beta(t)}^k \frac{T_j \rho_{jk} \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)} + \sum_{j=\beta(t)}^k \rho_{jk} \sigma_j(t) \int_t^{T_{\beta(t)-1}} \sigma_j(t, u) du.$$

46. (Spot-LIBOR-measure dynamics, LFM)

Using the same notation as in Exercise 45, define

$$B_k(t) = \frac{P(t, T_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} (1 + \tau_j F_j(T_{j-1}))} P(t, T_{\beta(t)-1}).$$

This can be interpreted as present value of a portfolio that invests all in a quantity X_i of T_i zero-coupon bonds at time T_{i-1} , starting with 1 at time $t = 0$. The measure \mathbb{Q}^t

associated with B_k is called spot LIBOR measure. Show that the dynamics of F_k under the spot LIBOR measure are:

$$dF_k(t) = \sigma_k(t)F_k(t) \sum_{j=\beta(t)}^k \frac{T_j \rho_{jk} \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_k(t)F_k(t)dW_k^t(t).$$