Welcome to Math 1215

6.3 / 6.4 / 6.5 / 6.6

7.1 / 7.2 / 7.3 / 7.5 / 7.6 / 7.7

8.1 / 8.2 / 8.3 / 8.4 / 8.5 / 8.6 / 8.8 / 8.9

10.2 / 10.3 / 10.4 / 10.5 / 10.6 / 10.7 / 10.8

11.1 / 11.2 / 11.3 / 11.4

12.1 / 12.2 / 12.3

Welcome to Math 1215

Spring 2022

Martin Bohner

Welcome to Math 1215

Lecture Professor: Dr. Bohner

Course Coordinators: Dr. Bohner and Mr. Runnion

GTAs: Daniel Kovach, Yumeng Wang, Youxin Yuan

There are 12 labs this semester.

You MUST attend the correct lab section!

Some Important Notes

Math 1215 Common Evening Exams

  5:00 – 5:50 PM, in person

  Thurs. Feb. 10, Thurs. Mar. 10, Thurs. Apr. 21

The exams are listed in the schedule of classes, and they are considered regular class meeting times.

You are responsible for working out any conflicts to ensure that you are present.

Missed exams will count as a zero.

Calculators are not allowed!

Some Important Notes

Common Final Exam

7:30 AM – 9:30 AM, Thursday, May 12, in person

The final exam is comprehensive.

Calculators are not allowed!

Each exam (regular and final) is 100 points.

The best three exams are counted.

Some Important Notes

If you have a documented disability and need accommodations in this course, you need to provide the paperwork by Monday January 31 so that arrangements can be made in time for Exam 1.

Electronic Devices

You must speak with me in advance before recording any class activity. It is a violation of University policy to distribute such recordings without my authorization and the permission of all others who are recorded.

Canvas

Math 1215 will be using Canvas this semester.

Visit umsystem.instructure.com to find:

- Syllabus / Calendar
- MyLab Math access
- Lecture slides
- Grades

Textbook and MyLab Math

This is an AutoAccess course.

You will see a charge on your student account.

This charge covers access to MyLab Math and a full electronic copy of the textbook, both of which are available via Canvas.

If you wish to purchase a loose-leaf (3-hole-punched) copy of the textbook at a discounted price, see the bookstore.

Warning: Access codes purchased from other sources may not work!

MyLab Math Assignments

- Daily homework
  - Includes material from that day’s lecture
  - Often includes review topics
- Weekly quizzes
  - Almost every Thursday, typically excluding exam weeks, breaks, and Week 1.
- Skills Check Quizzes
  - Checks prerequisite skills at the beginning of each chapter
- Skills Check Homework
  - Covers prerequisite skills not mastered on the Skills Check Quiz

Pivot to Online

If something occurs which causes some or all Math 1215 sections to suddenly pivot online, the syllabus provides details as to what will happen.

If necessary, exams will move to MyLab Math with ***no partial credit*** (but with calculators allowed).

Office Hours

Sections 302, 303, 307, 308:

Daniel Kovach (301 Rolla Bldg) M 2-4 pm

Sections 301, 305, 306, 310:

Yumeng Wang (105 Rolla Bldg) TTh 3:30-4:30 pm

Sections 304, 309, 311, 312:

Youxin Yuan (105 Rolla Bldg) MW 8-9 am

Martin Bohner (zoom) MWF 1-1:50 pm

Three Final Things

You are responsible for reading the rest of the syllabus.

If you have any questions, please contact me.

Good luck this semester!

$V = \pi z^2 a$

$V = Pi(z * z)a$

Section 6.3

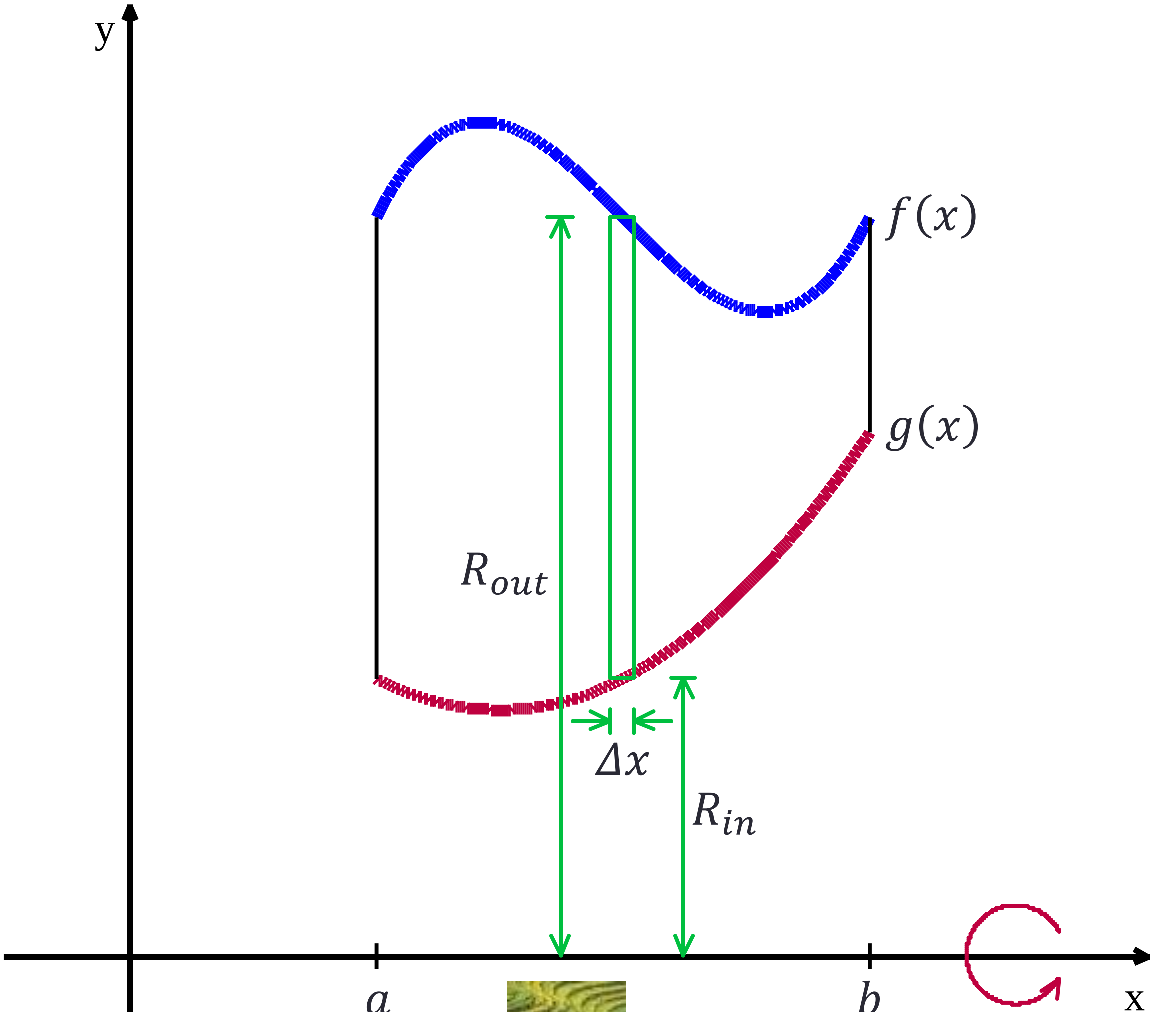
Volume by Slicing

Wednesday, January 19, 2022

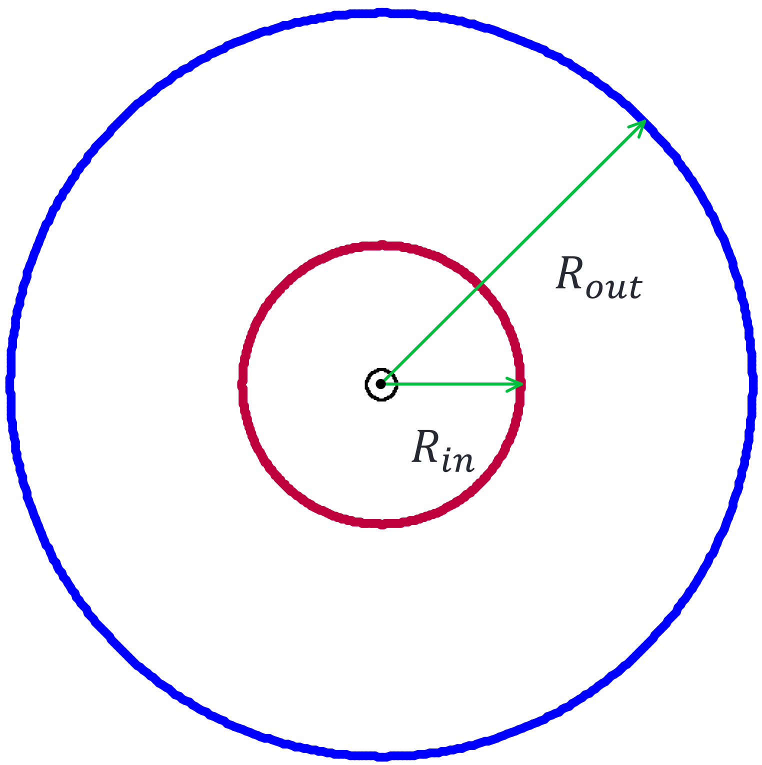
Solids of Revolution

If you take a well-defined region in a plane, identify a linear axis in the same plane, and rotate the region around the selected axis, it forms a three-dimensional solid of revolution.

Calculating Volumes: The Washer Method



Calculating Volumes: The Washer Method



Cross-sectional area

$$A(x) = \pi[R_{out}]^2 - \pi[R_{in}]^2$$

Volume of a cross-sectional washer

$$A(x)\Delta x$$

Volume of a solid of revolution

$$\int_a^b A(x)\mathrm{d}x$$

Example 6.3.1

Calculate the volume of the solid of revolution formed by revolving the region bounded by $f(x) = \sqrt{x}$ and $g(x) = x^2$ about the x-axis.

Example 6.3.2

Calculate the volume of the solid of revolution formed by revolving the region bounded by $y = x$, $y = 3$, and $x = 0$ about the line $y = 4$.

Example 6.3.3

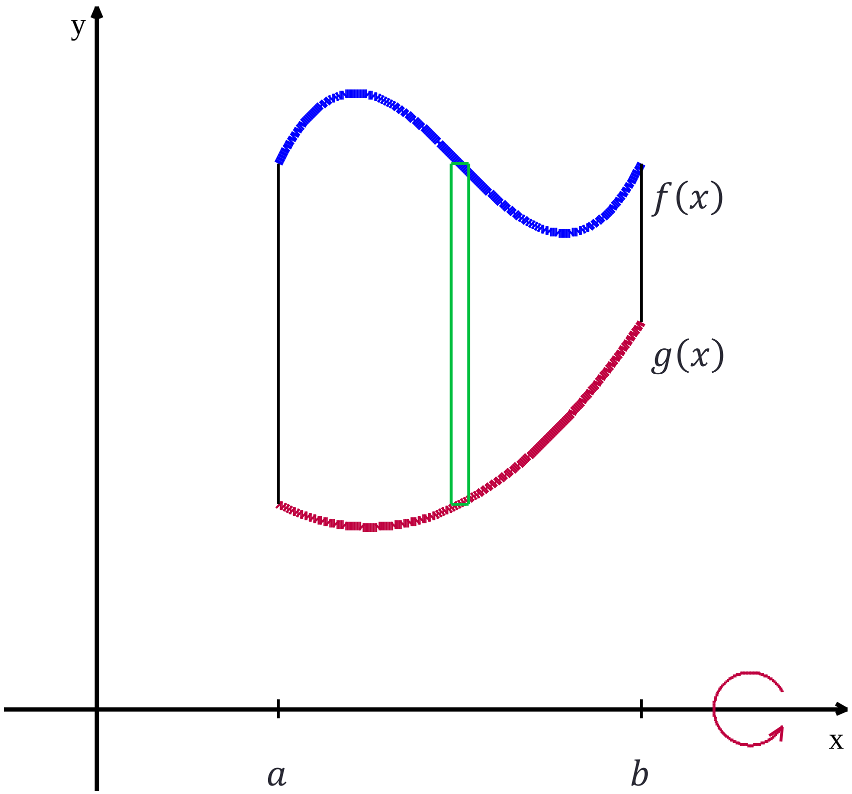
Calculate the volume of the solid of revolution formed by revolving the region bounded by $y = x^2 + 1$, $x = 1$, $x = 0$, and $y = 0$ about the y-axis.

Section 6.4

Volume by Shells

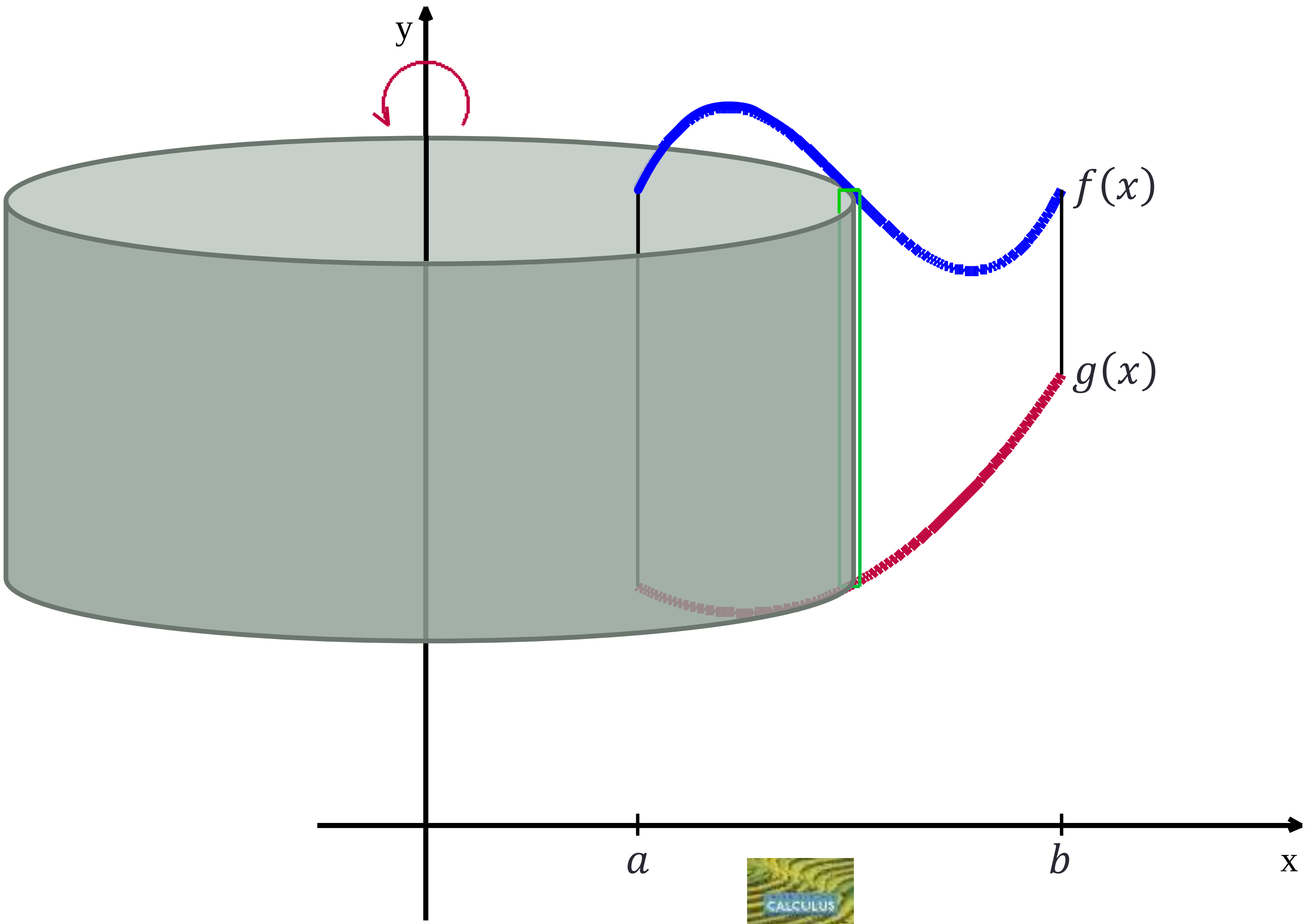
Friday, January 21, 2022

Calculating Volumes: The Washer Method

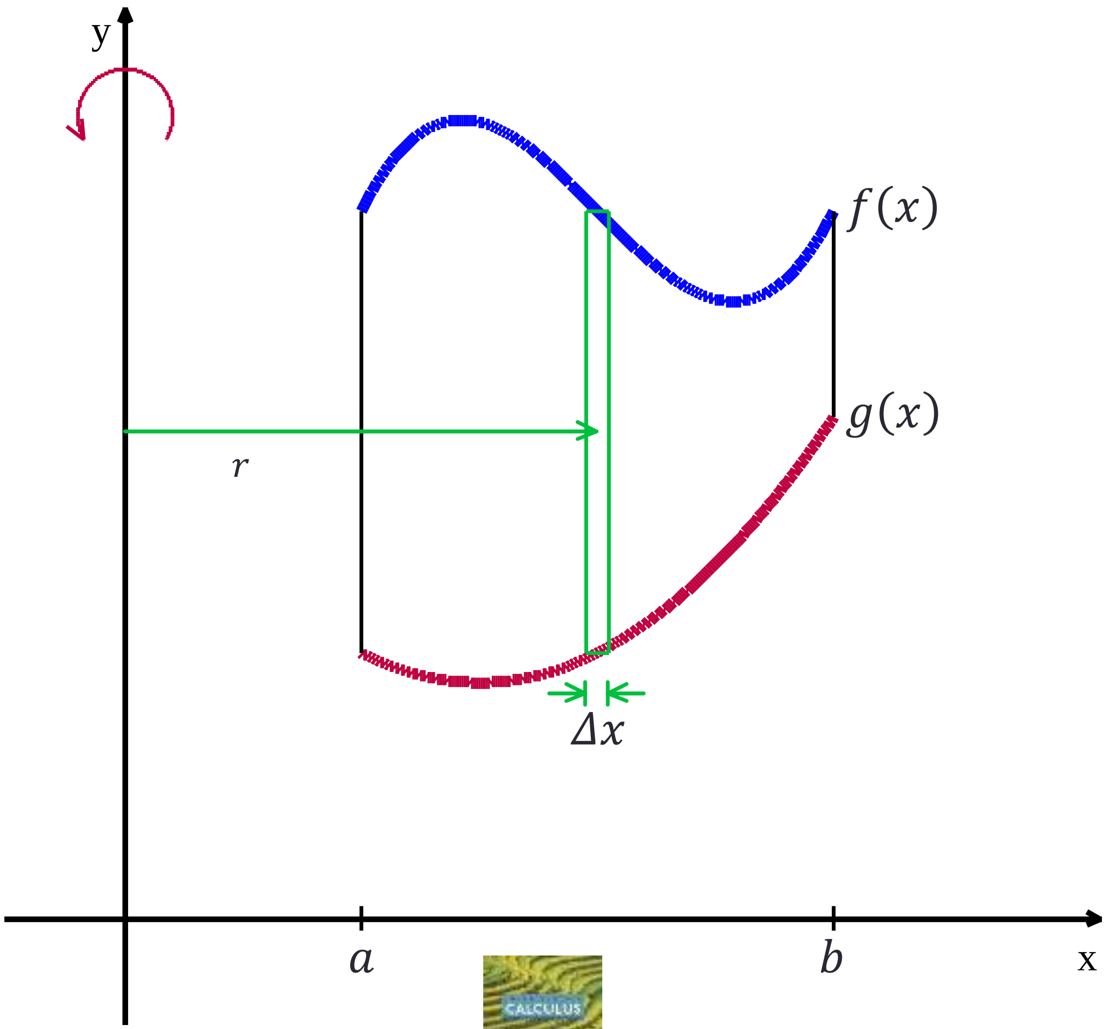


- Note that the rectangular slice is perpendicular to the axis of revolution
- Integration occurs along the axis of revolution
- Rotating this region about the y-axis is highly inconvenient using washers.

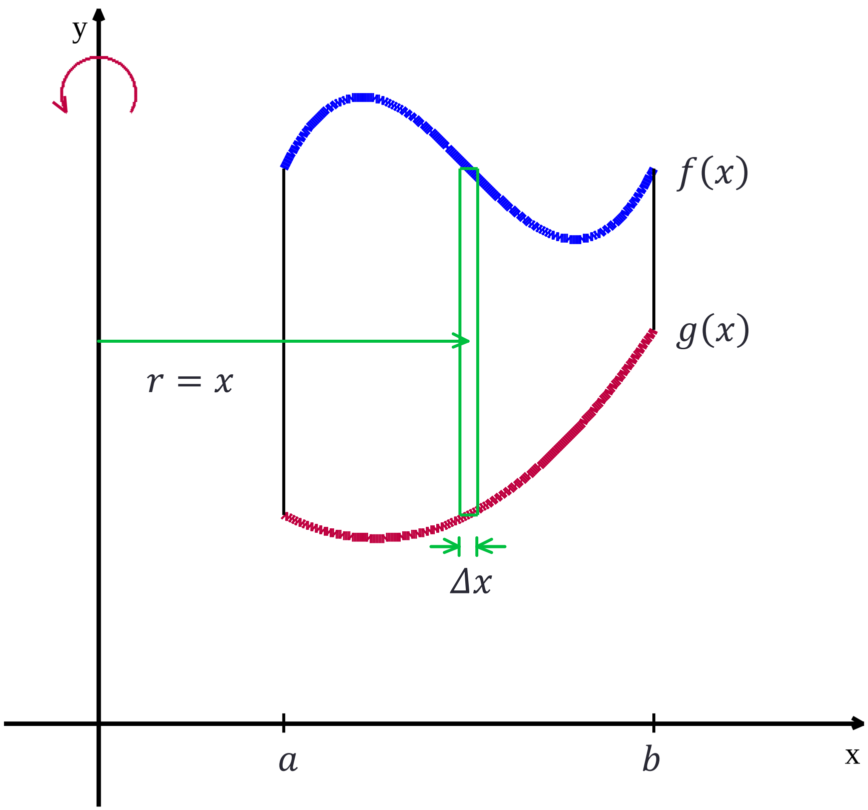
Calculating Volumes: The Shell Method



Calculating Volumes: The Shell Method



Calculating Volumes: The Shell Method



Surface area of a cylindrical shell

$$A(x) = 2\pi r h$$

$$A(x) = 2\pi x[f(x) - g(x)]$$

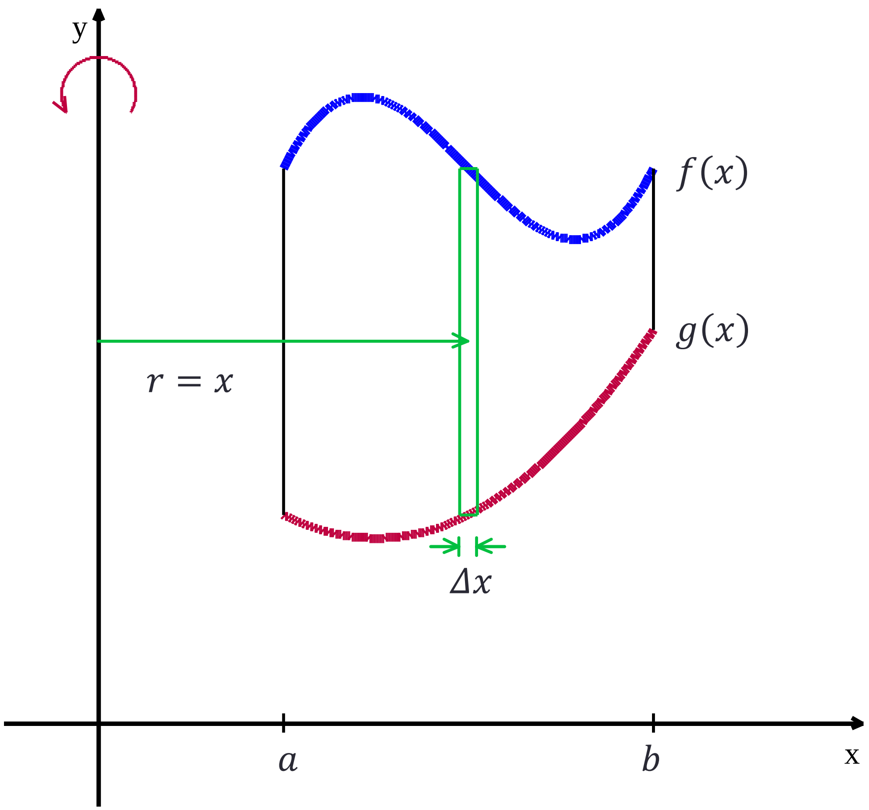
Volume of a cylindrical shell

$$V(x) = A(x)\Delta x$$

Volume of the solid

$$V = \int_a^b A(x)\mathrm{d}x$$

Calculating Volumes: The Shell Method



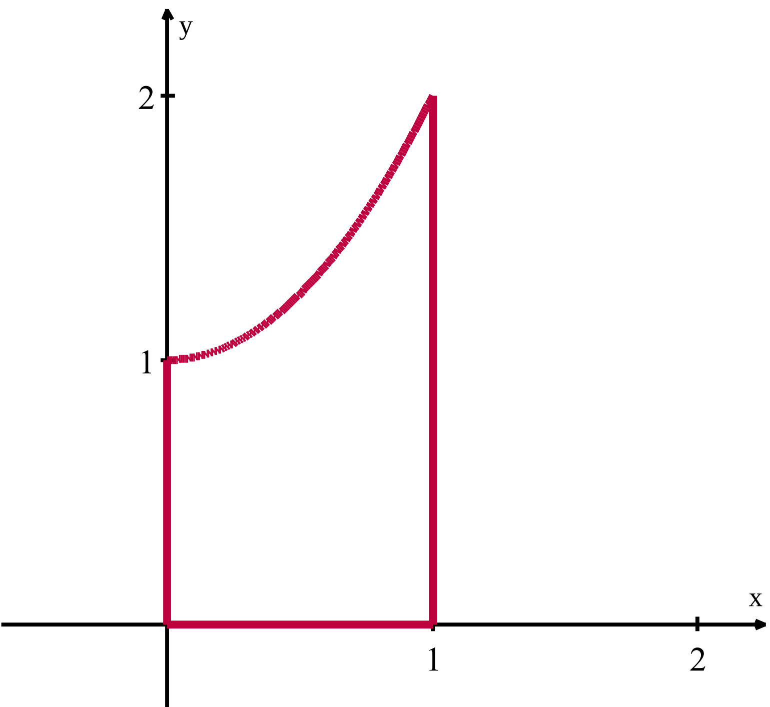
Volume of the solid

$$V = \int_a^b 2\pi r h \mathrm{d}x$$

- Note that the rectangular slice is parallel to the axis of revolution
- Integration occurs perpendicular to the axis of revolution

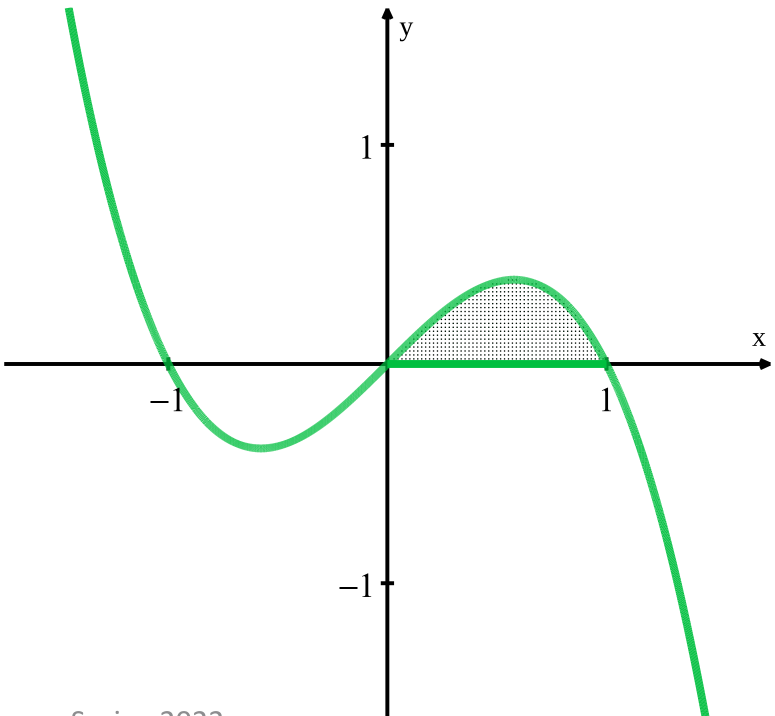
Example 6.4.1

Calculate the volume of the solid of revolution formed by revolving the region bounded by $y = x^2 + 1$, $x = 1$, $x = 0$, and $y = 0$ about the y-axis.



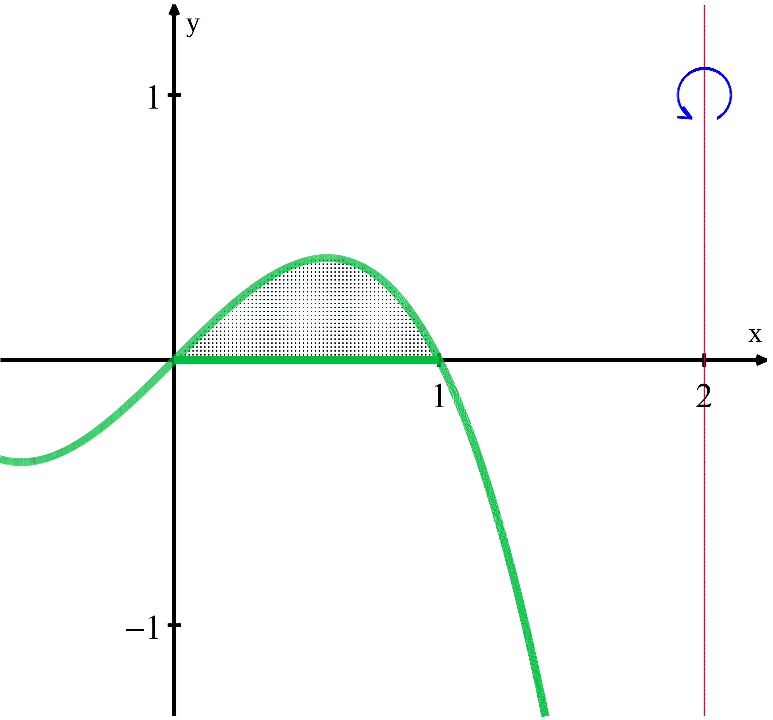
Example 6.4.2

Calculate the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis in the first quadrant about the y-axis.



Example 6.4.3

Calculate the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis in the first quadrant about the line $x = 2$.

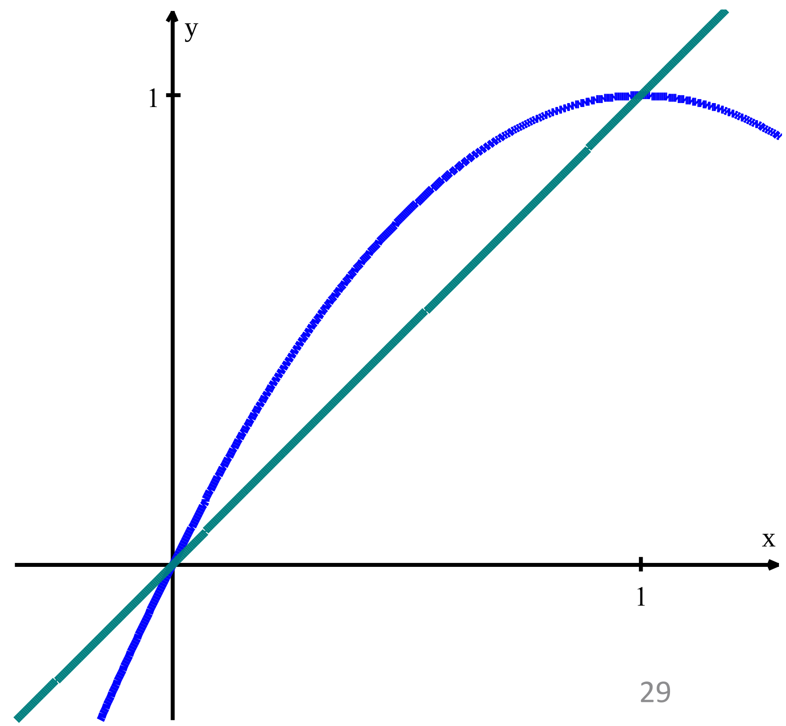


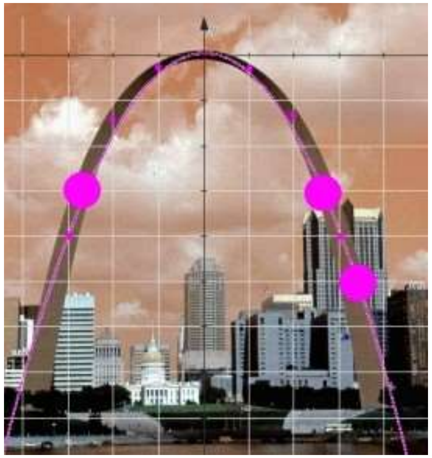
Example 6.4.4

Calculate the volume of the solid formed by revolving the region bounded by $f(x) = 2x - x^2$ and $g(x) = x$ about the x-axis using

a) the washer method.

b) the shell method.





Section 6.5

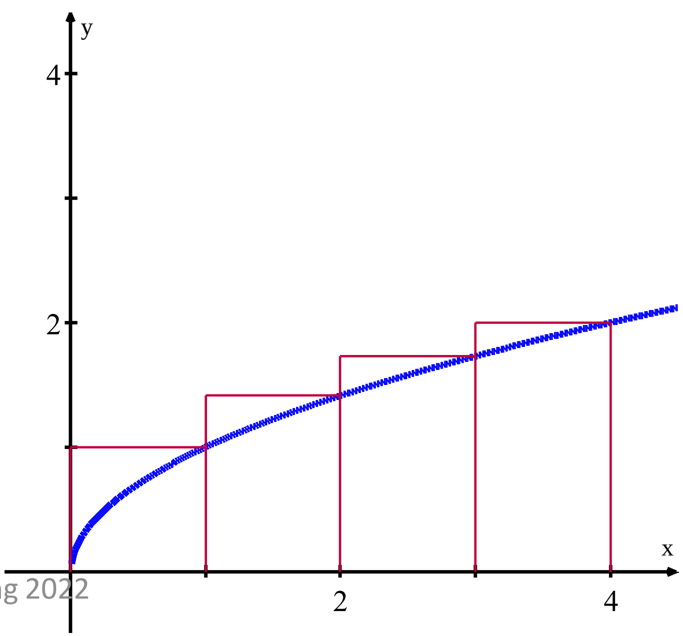
Length of Curves

Monday, January 24, 2022

Area vs. Arc Length

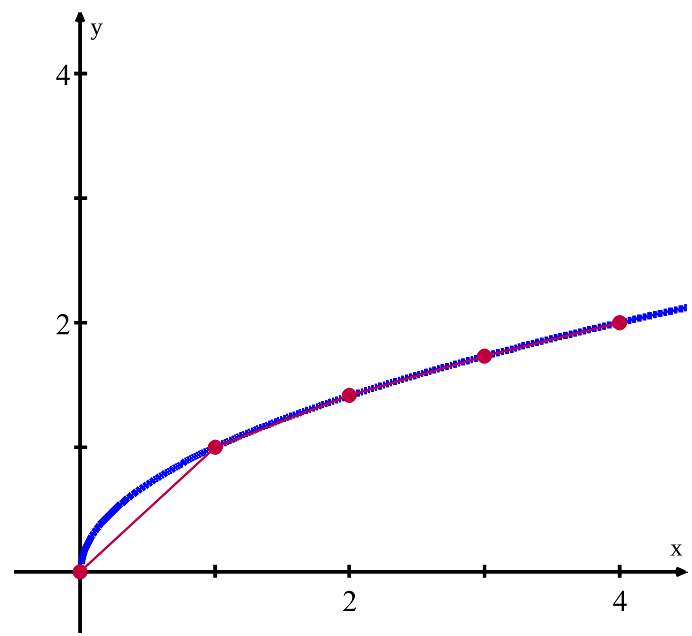
Calculus I: Area Under a Curve

Approximate the area under the curve $f(x) = \sqrt{x}$ on the interval $0 \leq x \leq 4$.

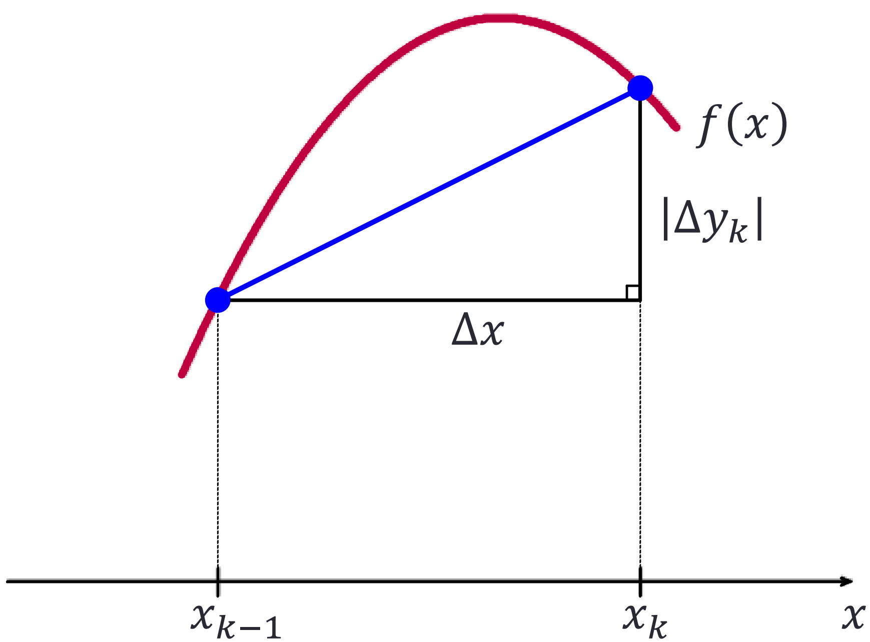


Calculus II: Arc Length

Approximate the arc length of the curve $f(x) = \sqrt{x}$ on the interval $0 \leq x \leq 4$.



Approximate Length of the kth Subinterval



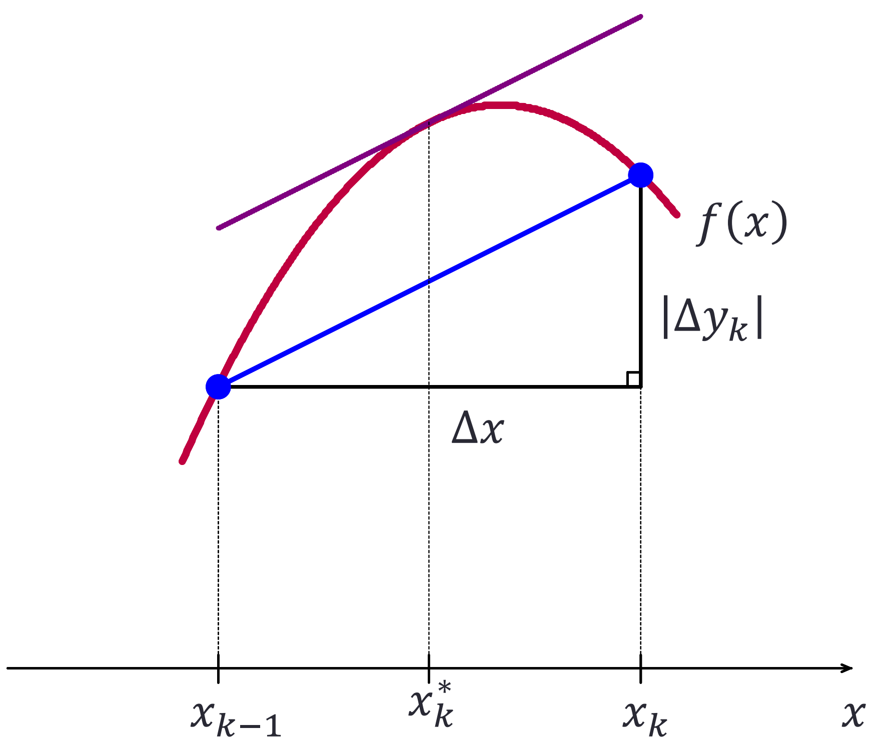
The line segment from $\left(x_{k-1}, f(x_{k-1})\right)$ to $\left(x_k, f(x_k)\right)$ has length

$$L = \sqrt{(\Delta x)^2 + |\Delta y_k|^2}$$

and slope

$$m = \frac{\Delta y_k}{\Delta x}$$

Approximate Length of the kth Subinterval



$$
\begin{aligned}
L &= \sqrt{(\Delta x)^2 + |\Delta y_k|^2} \\
&= \sqrt{(\Delta x)^2 + (\Delta y_k)^2} \\
&= \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2\right)} \\
&= \sqrt{\left(1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2\right)}\, \Delta x \\
&= \sqrt{\left(1 + \left(f'(x_k^*)\right)^2\right)}\, \Delta x
\end{aligned}
$$

The Arc Length Formula

If the curve $y = f(x)$ has continuous first derivatives on $[a, b]$, then the length of $y = f(x)$ on the interval $[a, b]$ is

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,\mathrm{d}x$$

Example 6.5.1

Find the arc length of the curve

$$y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$$

on the interval $[1,9]$.

Example 6.5.2

Set up an integral to find the arc length of

$$y = \cos x$$

on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Then, use technology to find an approximation.

The Arc Length Formula – Functions of y

If the curve $x = g(y)$ has continuous first derivatives on $[c, d]$, then the length of $x = g(y)$ on the interval $[c, d]$ is

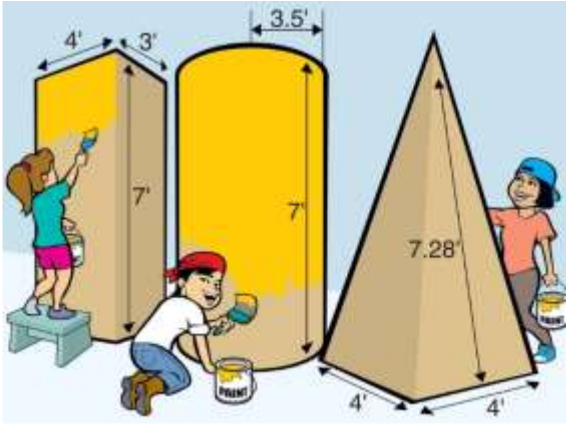
$$L = \int_{c}^{d} \sqrt{1 + \left(g'(y)\right)^2}\,\mathrm{d}y$$

Example 6.5.3

Find the arc length of the curve

$$x = \frac{1}{3}(y^2 + 2)^{3/2}$$

on the interval $0 \leq y \leq 4$.



Section 6.6

Surface Area

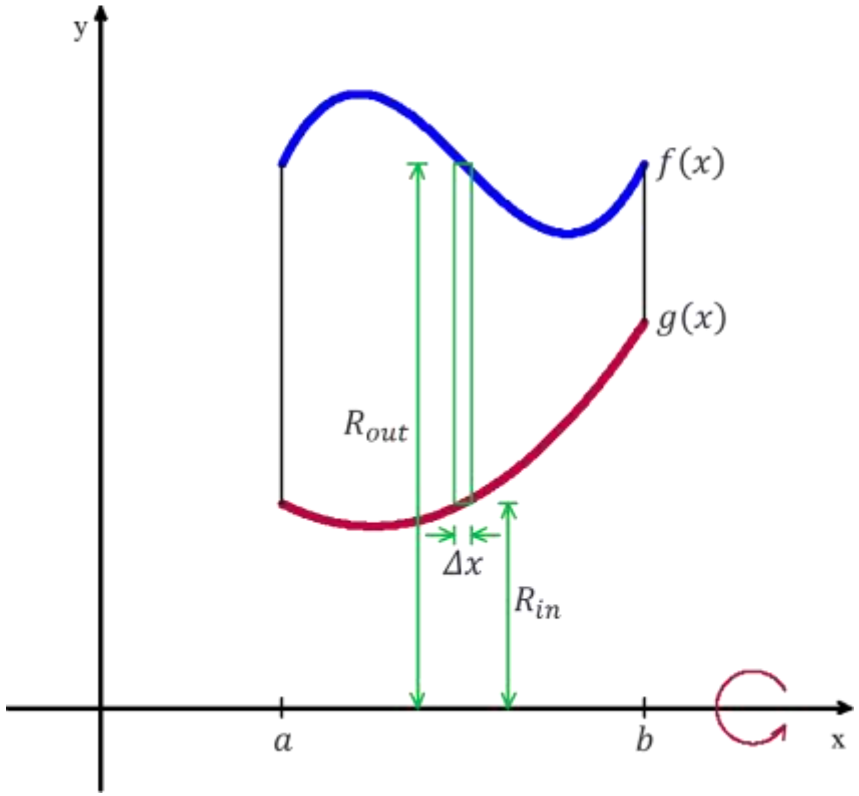
Wednesday, January 26, 2022

Surface Area

Goal for today:

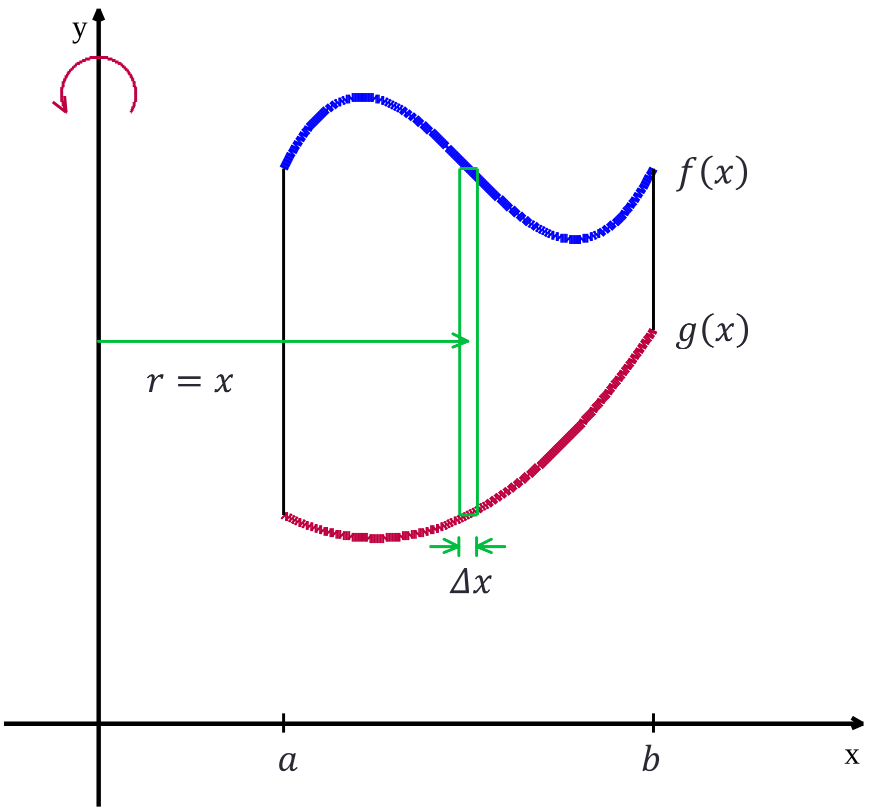
Calculate the surface area of a surface of revolution.

Review: Volume by Washers



$$V = \pi \int_a^b \left([R_{out}]^2 - [R_{in}]^2\right) \mathrm{d}x$$

Review: Volume by Shells



$$V = 2\pi \int_a^b r\big(f(x) - g(x)\big)\mathrm{d}x$$

or, equivalently

$$V = 2\pi \int_a^b rh\mathrm{d}x$$

Area of a Surface of Revolution

Let f be a nonnegative function with continuous first derivative on the interval $[a,b]$. The area of the surface formed by rotating the graph of $y=f(x)$ on the interval $[a,b]$ about the x-axis is

$$S=2\pi\int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\,\mathrm{d}x$$

Example 6.6.1

Find the area of the surface generated when the graph of $f(x) = x^3$ on the interval $[0,1]$ is rotated about the x-axis.

Example 6.6.2

Find the area of the surface generated when the graph of $y = \sqrt{4 - x^2}$ on the interval $[-1,1]$ is rotated about the x-axis.

Area of a Surface of Revolution

Let g be a nonnegative function with continuous first derivative on the interval $[c, d]$. The area of the surface formed by rotating the graph of $x = g(y)$ on the interval $[c, d]$ about the y-axis is

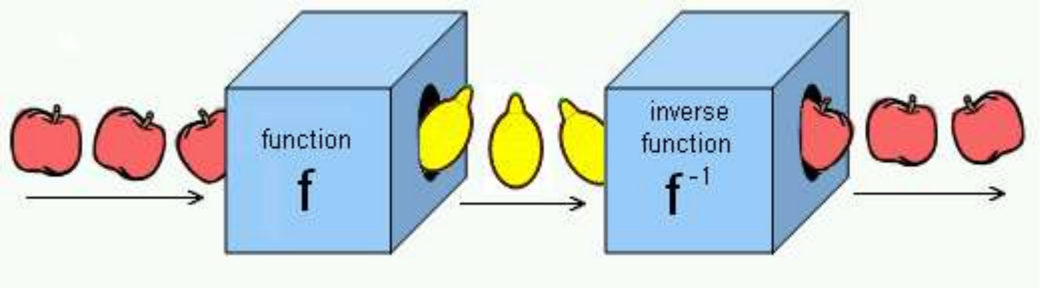
$$S = 2\pi \int_c^d g(y)\sqrt{1 + \left(g'(y)\right)^2}\,\mathrm{d}y$$

Example 6.6.3

Find the area of the surface generated when the graph of $y = 1 - \frac{x^2}{4}$ on the interval $0 \leq x \leq 2$ is rotated about the y-axis.

Example 6.6.4

Find the area of the surface generated when the graph of $f(x) = \frac{3}{2}x^{2/3}$ on the interval $1 \leq x \leq 8$ is rotated about the x-axis.



Section 7.1

Inverse Functions

Friday, January 28, 2022

One-to-One Functions

A function f is called one-to-one if, for any two values $x_1 \neq x_2$ in the domain of f, $f(x_1) \neq f(x_2)$.

In other words, a function f is called one-to-one if no two inputs produce the same output.

Identifying One-to-One Functions

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Details can be easily missed when looking at a graph

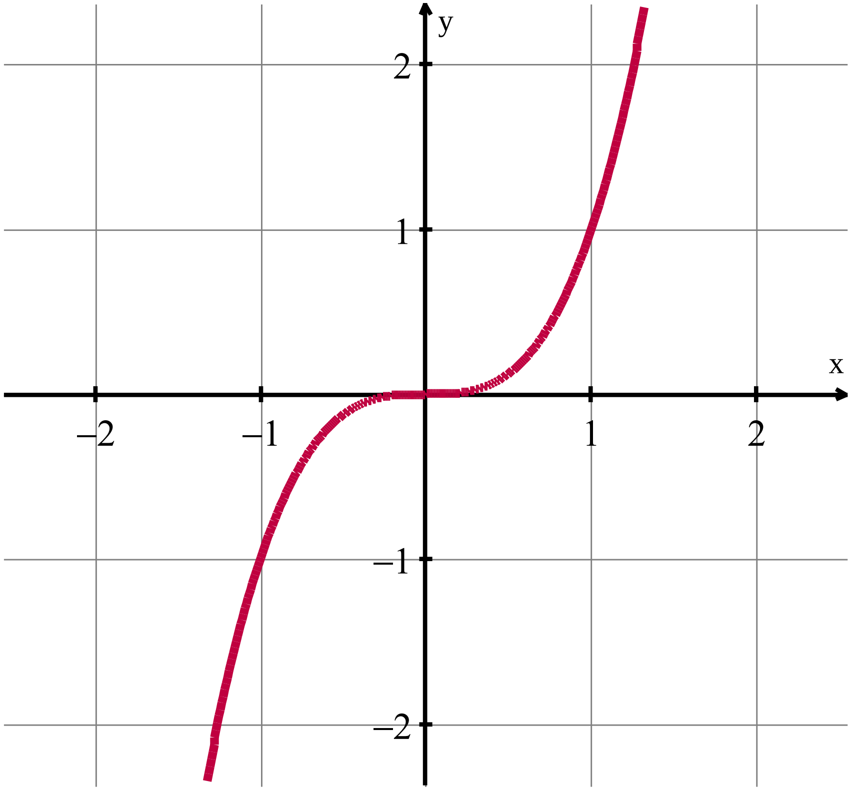
Calculus Techniques

If a function is *always* increasing or *always* decreasing,
then it is one-to-one.

Calculus can give us a conclusive method whenever our function f is continuous, differentiable, and $f'(x)$ is never zero.

Example 7.1.1

Determine whether $f(x) = x^3$ is a one-to-one function.



Recall: Increase and Decrease

Suppose f is continuous and differentiable on an open interval I.

If $f'(x) > 0$ on I, then f is increasing on I.

If $f'(x) < 0$ on I, then f is decreasing on I.

Inverse Functions

A function f is invertible if and only if it is one-to-one.

If f has domain A and range B, then its inverse function f^{-1} has domain B and range A.

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$

for any x in A and corresponding y in B.

Example 7.1.2

Consider the function f defined by the following table:

x	$f(x)$
1	2
2	6
3	4
4	-3
5	1

Calculate each quantity, if it exists:

$$f(2)$$

$$f^{-1}(2)$$

$$f(5)$$

$$f^{-1}(5)$$

Properties of Inverse Functions

If an invertible function f has domain A and range B, then

- $f^{-1}(f(x)) = x$ for every x in A

- $f(f^{-1}(x)) = x$ for every x in B

Finding an Inverse Function

Graphically

Given the graph of a one-to-one function f, reflect it about the line $y = x$ to obtain f^{-1}.

Symbolically

1. If relevant, remove function notation by replacing $f(x)$ with y.
2. Solve the equation for x in terms of y (if possible).
3. If the chosen variables have no clearly associated meaning, swap x and y and replace y with $f^{-1}(x)$.

Example 7.1.3

Consider the function $f(x) = \frac{2x+1}{x-1}$.

a) State the domain of f.

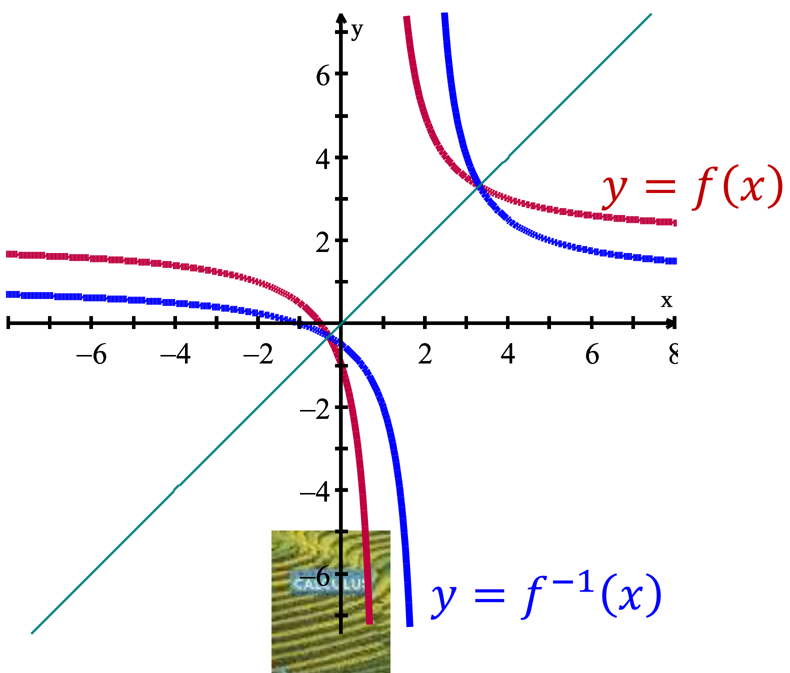
b) Show that f is one-to-one.

c) Calculate $f^{-1}(x)$ and state the domain and range of f^{-1}.

Example 7.1.3 (continued)

Consider the function $f(x) = \frac{2x+1}{x-1}$.

d) Sketch the graphs of f and f^{-1} on the same set of axes.



Continuity of Inverse Functions

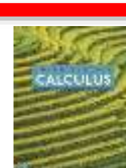
If f is continuous and invertible on an interval I, then its inverse function f^{-1} is also continuous on the corresponding interval consisting of all values $f(x)$ where x is in I.

Differentiability of Inverse Functions

Let f be a differentiable function which is invertible on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then the inverse function f^{-1} is differentiable at $y_0 = f(x_0)$. Specifically,

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$



Example 7.1.3 (continued)

Consider the function $f(x) = \frac{2x+1}{x-1}$.

e) Use the preceding theorem to find $(f^{-1})'(5)$.

f) Recall that $f^{-1}(x) = \frac{x+1}{x-2}$. Use this formula to find $(f^{-1})'(5)$.

Example 7.1.4

For $f(x) = x^5 + x^3 + 2x$, find $(f^{-1})'(4)$.

Example 7.1.5

Use the table to find each indicated derivative or state that the derivative cannot be determined.

x	-4	-2	0	2	4
$f(x)$	0	1	2	3	4
$f'(x)$	5	4	3	2	1

a) $f'\big(f(0)\big)$

b) $(f^{-1})'(0)$

c) $(f^{-1})'(1)$

d) $(f^{-1})'\big(f(4)\big)$

Section 7.2

The Natural Logarithmic and Exponential Functions

Monday, January 31, 2022

Exponential and Logarithmic Functions

In College Algebra, we

- Defined exponential functions based on growth rates
  - Typically focused on bases such as 2 and 10
- Defined logarithmic functions as the inverse of exponential functions
- Introduced lots of properties
- Introduced this funny number *e*

Logarithmic and Exponential Functions

In Calculus, we will

- Define the natural logarithm $\ln x$ as a definite integral.
  - Allows us to derive properties rather than simply state them
- Define the natural exponential e^x as the inverse of $\ln x$.
- Do some calculus with e^x and $\ln x$.
- Address bases other than e using these results.

The Natural Logarithm

The natural logarithm of a number $x > 0$ is

$$\ln x = \int_1^x \frac{\mathrm{d}t}{t}$$

The domain of $\ln x$ is $(0, \infty)$.

The range of $\ln x$ is $(-\infty, \infty)$.

Example 7.2.1

Find $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)$ using the integral definition of $\ln x$.

Recall: Fundamental Theorem of Calculus (Part I)

If f is continuous on $[a, b]$, then the function

$$A(x) = \int_a^x f(t)\mathrm{d}t$$

(where $a \leq x \leq b$) is continuous on $[a, b]$, differentiable on (a, b), and $A'(x) = f(x)$.

Example 7.2.2

Find each derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(-x)\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln|x|\right)$$

The Derivative of the Natural Logarithm

$$\frac{\mathrm{d}}{\mathrm{dx}}(\ln x) = \frac{1}{x} \qquad (x > 0)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln|x|) = \frac{1}{x} \qquad (x \neq 0)$$

Example 7.2.3

Find the derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln|\sec x|\right)$$

Integrals Involving Logarithms

Since we know

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln|x|) = \frac{1}{x}$$

we can conclude

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$

Example 7.2.4

Evaluate each integral.

$$\int \frac{\mathrm{d}x}{4-3x}$$

$$\int \tan x \, \mathrm{d}x$$

$$\int_0^3 \frac{2x-1}{x+1} \mathrm{d}x$$

Example 7.2.5

Use the integral definition of $\ln x$ to prove the following property.

$$\ln(xy) = \ln x + \ln y$$

Properties of Natural Logarithms

For $x > 0$, $y > 0$, and real values p, we have

$$\ln xy = \ln x + \ln y$$

$$\ln \frac{x}{y} = \ln x - \ln y$$

$$\ln x^p = p \ln x$$

Invertibility of $f(x) = \ln x$

By the Fundamental Theorem (Part I),
$f(x) = \ln x$ is continuous for $x > 0$.

Since $\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x} > 0$, we know $f(x) = \ln x$ is always increasing and thus one-to-one.

A function which is one-to-one and continuous on its domain is invertible.

Euler's Number

Euler's Number is the real number e which satisfies

$$\ln e = \int_1^e \frac{\mathrm{d}t}{t} = 1$$

It can be shown that $e \approx 2.71828$

The Inverse of $f(x) = \ln x$

Recall that $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$ for appropriate values of x.

By the laws of logarithms, we have

$$\ln(e^x) = x\ln e = x.$$

Thus, the inverse of $f(x) = \ln x$ is

$$f^{-1}(x) = e^x$$

$y = e^x$ is the natural exponential function.

Properties of the Natural Exponential

$$e^{\ln x} = x \text{ for all } x > 0$$

$$\ln(e^x) = x \text{ for all } x$$

$$e^{x+y} = e^x e^y$$

$$e^{x-y} = \frac{e^x}{e^y}$$

$$(e^x)^r = e^{rx}$$

Example 7.2.6

Calculate the derivative of $g(x) = e^x$.

Hint: Use $g = f^{-1}$ with $f(x) = \ln x$.

Derivatives and Integrals of e^x

$$\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x$$

$$\int e^x \mathrm{d}x = e^x + C$$

Example 7.2.7

Calculate each quantity.

$$\int e^x(4+e^x)^5\,\mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(\sec^4 x\tan^2 x\right)\right]$$

Logarithmic Differentiation

Goal: Use the properties of logarithms to help simplify the process of differentiating functions involving products, quotients, and/or powers.

Logarithmic Differentiation

Procedure:

1) Take natural logarithms of both sides of the equation $y = f(x)$.
2) Differentiate both sides implicitly with respect to x.
3) Solve the resulting equation for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Example 7.2.8

Use logarithmic differentiation to differentiate.

$$f(x) = \frac{x^8 \cos^3 x}{\sqrt{x-1}}$$

Example 7.2.9

Evaluate each integral given that $cabin > 0$.

$$\int_{1}^{cabin} \frac{\mathrm{d}x}{x}$$

$$\int \frac{\mathrm{d}cabin}{cabin}$$

Section 7.3

Logarithmic and Exponential Functions with other Bases

Wednesday, February 2, 2022

General Exponential Functions

If $a > 0$, then we define

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$

Example 7.3.1

In Section 7.2, we showed that

$$\ln ab = \ln a + \ln b$$

Use this property of logarithms to prove the following property of exponents.

$$(ab)^x = a^x b^x$$

Properties of General Exponentials

If a and b are both positive and x and y are both real, then

$$a^{x+y} = a^x a^y$$

$$a^{x-y} = \frac{a^x}{a^y}$$

$$(a^x)^y = a^{xy}$$

$$(ab)^x = a^x b^x$$

Example 7.3.2

Find the derivative of $f(x) = a^x$.

Derivatives and Integrals of $f(x) = a^x$

Let $a > 0$ and $a \neq 1$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}[a^x] = a^x \ln a$$

and

$$\int a^x \mathrm{d}x = \frac{a^x}{\ln a} + C$$

Example 7.3.3

Calculate the derivative and the general antiderivative of each function.

$$f(x) = 3^x - x^3$$

$$g(x) = x^2 10^{x^3}$$

General Logarithmic Functions

The inverse of the general exponential function with base $a \neq 1$ is the logarithmic function with base a.

Notation: $\log_a x$

The properties of $\log_a x$ correspond directly to those of $\ln x$.

Change of Base Formula

For any positive base $a \neq 1$,

$$\log_a x = \frac{\ln x}{\ln a}.$$

For any positive bases $a \neq 1$ and $b \neq 1$,

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Example 7.3.4

Provided $a > 0$ and $a \neq 1$, use the change-of-base formula to find

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_a x]$$

Derivatives of General Logarithms

Provided $a > 0$ and $a \neq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_a x] = \frac{1}{x \ln a}$$

for $x > 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_a |x|] = \frac{1}{x \ln a}$$

for $x \neq 0$.

Example 7.3.5

Find the derivative of each function.

$$f(x) = 2x\log_{10}\sqrt{x}$$

$$g(x) = \log_2 \frac{8}{\sqrt{x+1}}$$

The General Power Rule

For any real number p and for $x > 0$,

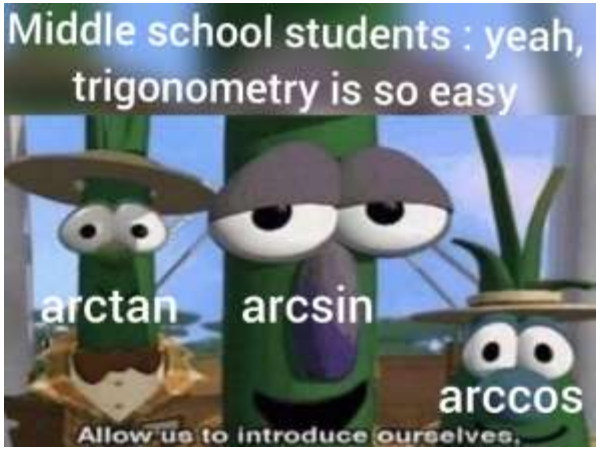
$$\frac{\mathrm{d}}{\mathrm{d}x}[x^p] = px^{p-1}$$

Example 7.3.6

Calculate the derivative of each function.

$$f(x) = e^x x^e$$

$$g(x) = x^{\cos x}$$



Section 7.5

Inverse Trigonometric Functions

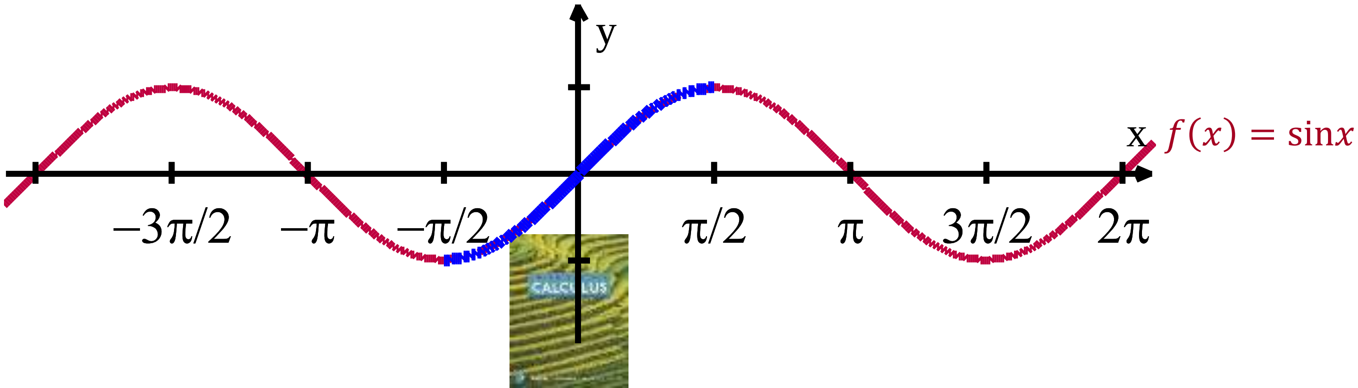
Friday, February 4, 2022

The Inverse Sine Function

$f(x) = \sin x$ is not invertible, unless the domain is restricted.

Restricted Domain: $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

The inverse of $f(x) = \sin x$ on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the inverse sine (or arcsine) function.



The Inverse Sine Function

Notation: $\sin^{-1}(x) = \arcsin(x)$

Domain of $\sin^{-1}(x)$: $[-1,1]$ range of $\sin x$

Range of $\sin^{-1}(x)$: $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ restricted domain of $\sin x$

Cancellation Properties:

$$\sin^{-1}(\sin x) = x \text{ provided } \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \text{ provided } -1 \leq x \leq 1$$

Example 7.5.1

Calculate the requested quantity.

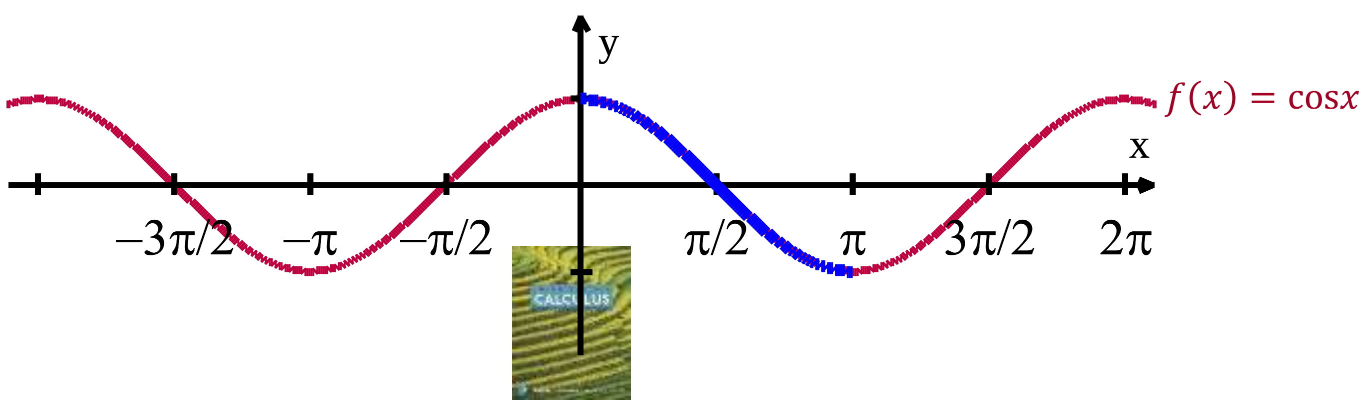
$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

The Inverse Cosine Function

$f(x) = \cos x$ is not invertible, unless the domain is restricted.

Restricted Domain: $[0, \pi]$

The inverse of $f(x) = \cos x$ on $[0, \pi]$ is called the inverse cosine (or arccosine) function.



The Inverse Cosine Function

Notation: $\cos^{-1}(x) = \arccos(x)$

Domain of $\cos^{-1}(x)$: $[-1,1]$ range of $\cos x$

Range of $\cos^{-1}(x)$: $[0, \pi]$ restricted domain of $\cos x$

Cancellation Properties:

$$\cos^{-1}(\cos x) = x \text{ provided } 0 \leq x \leq \pi$$

$$\cos(\cos^{-1} x) = x \text{ provided } -1 \leq x \leq 1$$

Example 7.5.2

Simplify the expression by rewriting it as an algebraic expression.

$$\cos(\sin^{-1} x)$$

The Inverse Tangent Function

$f(x) = \tan x$ is not invertible, unless the domain is restricted.

Restricted Domain: $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

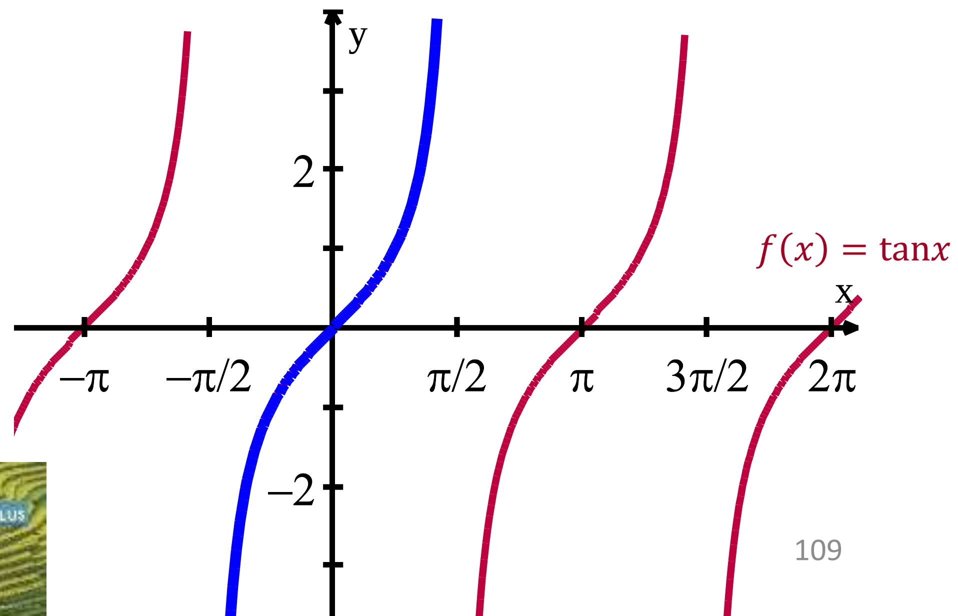
The inverse of

$f(x) = \tan x$ on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

is called the

inverse tangent

(or arctangent) function.



The Inverse Tangent Function

Notation: $\tan^{-1}(x) = \arctan(x)$

Domain of $\tan^{-1}(x)$: $\mathbb{R}$ range of $\tan x$

Range of $\tan^{-1}(x)$: $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ restricted domain of $\tan x$

Cancellation Properties:

$$\tan^{-1}(\tan x) = x \text{ provided } \frac{-\pi}{2} < x < \frac{\pi}{2}$$

$$\tan(\tan^{-1} x) = x \text{ for any real } x$$

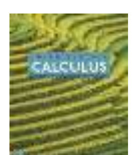
Example 7.5.3

Simplify the expression by rewriting it as an algebraic expression.

$$\cos(\tan^{-1} x)$$

Example 7.5.4

Find the derivative of $g(x) = \sin^{-1} x$.



Derivatives of Inverse Trig Functions

For $-1 < x < 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}x] = \frac{-1}{\sqrt{1-x^2}}$$

You should memorize $\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}x]$, and know the relationship between these two.

Example 7.5.5

Find the derivative of $g(x) = \tan^{-1} x$.

Derivatives of Inverse Trig Functions

For $-\infty < x < \infty$,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cot^{-1}x] = \frac{-1}{1+x^2}$$

You should memorize $\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x]$, and know the relationship between these two.

Derivatives of Inverse Trig Functions

For $|x| > 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\csc^{-1} x] = \frac{-1}{|x|\sqrt{x^2 - 1}}$$

You should know that these exist, but do not memorize them!

Example 7.5.6

Find the derivative of each function. Simplify where possible.

$$f(x) = \sin^{-1}\left(e^{\sin x}\right)$$

$$g(x) = \sin\left(\tan^{-1}(\ln x)\right)$$

$$h(x) = \cos(\sin^{-1} x)$$

Inverse Trigonometric Antiderivatives

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

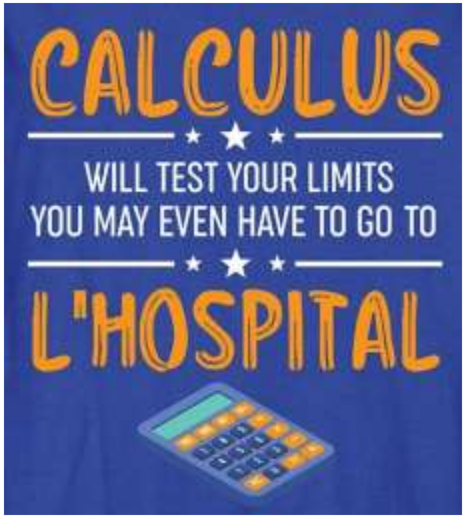
$$\int \frac{\mathrm{d}x}{1+x^2} = \tan^{-1} x + C$$

Example 7.5.7

Evaluate each integral.

$$\int \frac{\mathrm{d}x}{x^2+9}$$

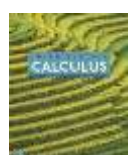
$$\int_0^3 \frac{\mathrm{d}x}{\sqrt{36-x^2}}$$



Section 7.6

L’Hôpital’s Rule

Monday, February 7, 2022



Indeterminate Forms

If the application of direct substitution and/or the basic laws of limits to a function results in a form which does not uniquely define the limit or even guarantee its existence, we are said to have encountered an indeterminate form, e.g.,

$$\underbrace{\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty}_{\text{We explored these four indeterminate forms in Calculus I}},\ \underbrace{0^0,\ \infty^0,\ 1^\infty}_{\text{We will address these today}}$$

L'Hôpital's Rule

Suppose f and g are both differentiable on an open interval containing c (except possibly at c). Further suppose that g is nonzero on the interval (except possibly at c). If

$$\lim_{x\to c}\frac{f(x)}{g(x)}$$

produces the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite).

WARNING!

Please note the distinct difference between the Quotient Rule – where we are taking the derivative of an entire quotient – and L'Hôpital's Rule – where we are taking the derivative of the numerator and denominator separately!

Example 7.6.1

Evaluate each of the following limits.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

L'Hôpital's Rule for Limits at Infinity

Suppose $f(x)$ and $g(x)$ are both differentiable for sufficiently large values of x. Further suppose that g is not identically zero for such values of x. If

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}$$

produces the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

(Similar results hold for limits approaching negative infinity.)

Example 7.6.2

Evaluate each of the following limits.

$$\lim_{x \to \infty} \frac{x^3}{x+2}$$

$$\lim_{x \to \infty} \frac{x^3}{e^{x/2}}$$

Example 7.6.3

Evaluate each limit using L'Hôpital's Rule.

$$\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$$

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$$

Example 7.6.4

Evaluate the limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.$$

The Number *e* as a Limit

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$$

If we let $n = \frac{1}{x}$, then we obtain the equivalent form

$$e = \lim_{n\to 0}(1+n)^{1/n}$$

The Indeterminate Forms 0^0, ∞^0, and 1^∞

If $\lim_{x \to a} f(x)$ results in the indeterminate form 0^0, ∞^0, or 1^∞,

1. Begin by writing $y = \lim_{x \to a} f(x)$
2. Take the natural logarithm of both sides
$$\ln y = \ln\left(\lim_{x \to a} f(x)\right) = \lim_{x \to a} \ln\big(f(x)\big)$$
3. Evaluate the limit using properties of logs
4. Solve for y

Example 7.6.5

Evaluate the limit

$$\lim_{x \to 1^+} (\ln x)^{x-1}.$$

Section 7.7

Hyperbolic Functions

Friday, February 11, 2022

A “Complex” View of Trigonometry

Euler’s Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Replacing θ with $-\theta$ and using the symmetry of sine and cosine, we obtain an alternate version of Euler’s formula:

$$e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta)$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

A "Complex" View of Trigonometry

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

By adding, we obtain $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, i.e.,

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$$

By subtracting, we obtain $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$, i.e.,

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$$

The Hyperbolic Functions

Hyperbolic Functions

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Corresponding Trigonometric Functions

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

The Hyperbolic Functions

Hyperbolic Functions

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

More Hyperbolic Functions

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\coth x = \frac{1}{\tanh x}$$

Symmetry

Hyperbolic Functions

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\tanh(-x) = -\tanh x$$

Corresponding Trigonometric Functions

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

Identities

Hyperbolic Functions	**Corresponding Trigonometric Functions**
$\cosh^2 x - \sinh^2 x = 1$	$\cos^2\theta + \sin^2\theta = 1$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$1 + \tan^2\theta = \sec^2\theta$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\cot^2\theta + 1 = \csc^2\theta$

Example 7.7.1

Prove the identity

$$1 - \tanh^2 x = \operatorname{sech}^2 x.$$

Example 7.7.2

Use the definition of the hyperbolic sine to find its derivative.

Derivatives of Hyperbolic Functions

Memorize These

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh x] = \cosh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh x] = \sinh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh x] = \mathrm{sech}^2 x$$

Know that These Exist

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{csch} x] = -\mathrm{csch} x \coth x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{sech} x] = -\mathrm{sech} x \tanh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\coth x] = -\mathrm{csch}^2 x$$

Example 7.7.3

Calculate the derivative of each function.

$$f(x) = \frac{\cosh(3x)}{x^2}$$

$$g(x) = \ln(\coth(3x))$$

Example 7.7.4

Calculate each integral.

$$\int_{0}^{\ln 3} \sinh^2 x \cosh x \, dx$$

$$\int \operatorname{sech}(2x) \tanh(2x) \, dx$$

Example 7.7.5

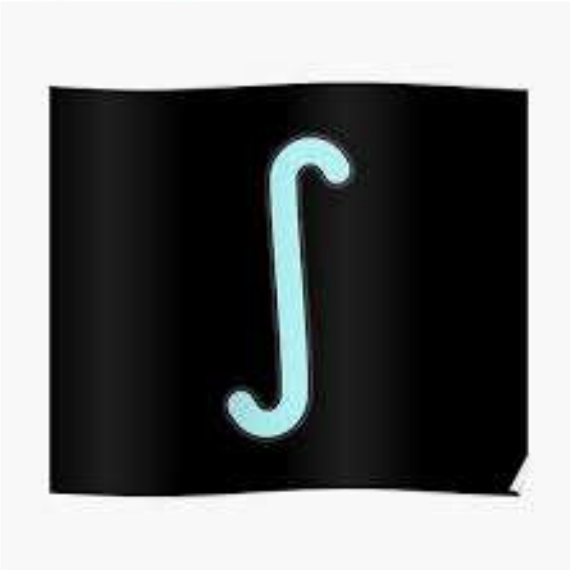
Calculate the integral

$$\int \tanh x \, \mathrm{d}x \,.$$

Inverse Hyperbolic Functions

There are inverse hyperbolic functions.

We will not cover them.



Section 8.1

Integration Techniques: Basic Approaches

Monday, February 14, 2022

Basic Integration Formulas

The Power Rule:

$$\int x^p \, \mathrm{d}x = \frac{x^{p+1}}{p+1} + C, \qquad p \neq -1$$

Basic Integration Formulas

Logarithmic and Exponential Integrals

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$

$$\int e^{x}\mathrm{d}x = e^{x} + C$$

Basic Integration Formulas

Trigonometric Integrals

$$\int \cos x \, \mathrm{d}x = \sin x + C \qquad \int \csc x \cot x \, \mathrm{d}x = -\csc x + C$$

$$\int \sin x \, \mathrm{d}x = -\cos x + C \qquad \int \sec x \tan x \, \mathrm{d}x = \sec x + C$$

$$\int \sec^2 x \, \mathrm{d}x = \tan x + C \qquad \int \csc^2 x \, \mathrm{d}x = -\cot x + C$$

Basic Integration Formulas

You will be expected to know all of the Basic Integration Formulas presented thus far on all subsequent exams in this course.

Slightly Less Basic Integration Formulas

More Trigonometric Integrals

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

Example 8.1.1

Compute the integral

$$\int \sec x \, \mathrm{d}x.$$

Slightly Less Basic Integration Formulas

More Trigonometric Integrals

$$\int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, \mathrm{d}x = \ln|\csc x - \cot x| + C$$

Slightly Less Basic Integration Formulas

Inverse Trigonometric Integrals

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \sin^{-1}x + C$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \tan^{-1}x + C$$

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C$$

Slightly Less Basic Integration Formulas

The Slightly Less Basic Integration Formulas will be provided on all subsequent exams in this course.

Example 8.1.2

Evaluate each integral.

$$\int_{3}^{7} (t-6)\sqrt{t-3}\,\mathrm{d}t$$

$$\int \frac{x+2}{x^2+4}\,\mathrm{d}x$$

Example 8.1.3

Evaluate each integral.

$$\int \frac{\mathrm{d}\theta}{1 - \csc\theta}$$

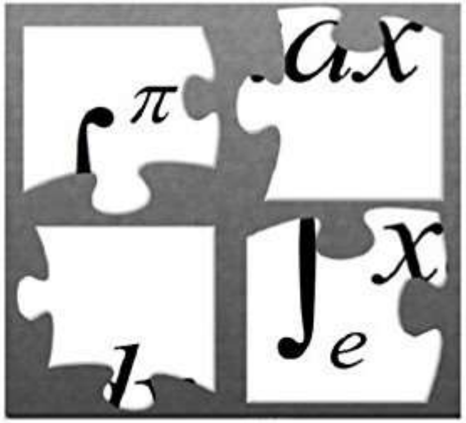
$$\int_0^2 \frac{x}{x^2 + 4x + 8}\,\mathrm{d}x$$

Example 8.1.4

Evaluate the integral

$$\int \frac{\mathrm{d}y}{y^{-1} + y^{-3}}.$$

Integration By Parts



Section 8.2

Integration by Parts

Wednesday, February 16, 2022

Integration by Parts

Goal: Evaluate integrals of certain products that cannot be handled with methods we already know.

Basic Idea: Develop the method by conveniently rearranging the Product Rule for Derivatives.

Rearranging the Product Rule

The Product Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Integrate both sides with respect to x, obtaining

$$f(x)g(x) = \int \big(f'(x)g(x) + f(x)g'(x)\big)\mathrm{d}x$$

Split the integral, yielding

$$f(x)g(x) = \int \big(f'(x)g(x)\big)\mathrm{d}x + \int \big(f(x)g'(x)\big)\mathrm{d}x$$

Rearranging the Product Rule

$$f(x)g(x) = \int \big(f'(x)g(x)\big)\mathrm{d}x + \int \big(f(x)g'(x)\big)\mathrm{d}x$$

Rearranging the terms yields

$$\int \big(f(x)g'(x)\big)\mathrm{d}x = f(x)g(x) - \int \big(f'(x)g(x)\big)\mathrm{d}x$$

Let $u = f(x)$ and $v = g(x)$, then $\mathrm{d}u = f'(x)\mathrm{d}x$ and $\mathrm{d}v = g'(x)\mathrm{d}x$, allowing us to write

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

Integration by Parts

If u and v are differentiable functions, then

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$$

Example 8.2.1

Evaluate each integral.

$$\int te^{6t}\mathrm{d}t$$

$$\int x^9 \ln x \, \mathrm{d}x$$

Example 8.2.2

Evaluate each integral.

$$\int \ln x \, \mathrm{d}x$$

$$\int_0^{\pi/2} x \cos(2x) \, \mathrm{d}x$$

Example 8.2.3

Evaluate each integral.

$$\int t^2 e^{-t} \mathrm{d}t$$

$$\int e^x \sin x \, \mathrm{d}x$$

Section 8.3

Trigonometric Integrals

Friday, February 18, 2022

Integrals Involving Sine and Cosine

When evaluating integrals of the form

$$\int \sin^m x \cos^n x \mathrm{d}x$$

three general approaches apply.

1. If the integrand can be rewritten so that $\mathrm{d}u = \cos x \, \mathrm{d}x$ remains and everything else is in terms of $\sin x$, we let $u = \sin x$.
   This works when n is odd and positive and m is real.

Integrals Involving Sine and Cosine

When evaluating integrals of the form

$$\int \sin^m x \cos^n x \mathrm{d}x$$

three general approaches apply.

2. If the integrand can be rewritten so that $-\mathrm{d}u = \sin x \, \mathrm{d}x$ remains and everything else is in terms of $\cos x$, we let $u = \cos x$.
   This works when m is odd and positive and n is real.

Integrals Involving Sine and Cosine

When evaluating integrals of the form

$$\int \sin^m x \cos^n x dx$$

three general approaches apply.

3. If neither $u = \sin x$ or $u = \cos x$ work directly, use the power reducing identities to rewrite the integrand.
   This is necessary when both m and n are even and nonnegative.

Power Reducing Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

These identities, which will be provided on exams, come from the double-angle formulas for cosine.

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

Example 8.3.1

Evaluate the integrals.

$$\int \sin^3 x \cos^5 x \,\mathrm{d}x$$

$$\int \sin^4(6x)\,\mathrm{d}x$$

Integrals Involving Secant and Tangent

When evaluating integrals of the form

$$\int \sec^m x \tan^n x \, dx$$

two general approaches apply.

1. If the integrand can be rewritten so that $du = \sec^2 x dx$ remains and everything else is in terms of $\tan x$, we let $u = \tan x$.
   This works when m is even and positive and n is real.

Integrals Involving Secant and Tangent

When evaluating integrals of the form

$$\int \sec^m x \tan^n x dx$$

two general approaches apply.

2. If the integrand can be rewritten so that $\mathrm{d}u = \sec x \tan x \,\mathrm{d}x$ remains and everything else is in terms of $\sec x$, we let $u = \sec x$. This works when n is odd and positive and m is real.

Example 8.3.2

Evaluate the integrals.

$$\int \sec^2 x \tan x \, dx$$

$$\int \sec^5 x \tan^3 x \, dx$$

Example 8.3.3

Evaluate the integrals.

$$\int \sec^3 x \, dx$$

$$\int_0^{\pi/4} \tan^4 x dx$$

More Sine and Cosine Integrals

To evaluate integrals of the forms

$$\int \sin mx \cos nx \, dx$$

$$\int \sin mx \sin nx \, dx$$

$$\int \cos mx \cos nx \, dx$$

we can use product-to-sum identities.

Product-to-Sum Identities

$$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

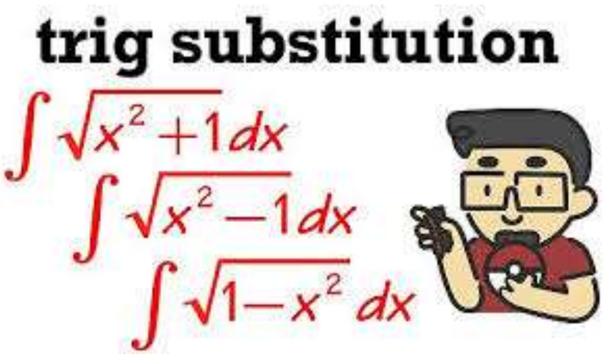
$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

These identities will be provided on exams.

Example 8.3.4

Evaluate the integral.

$$\int_{-\pi}^{\pi} \sin(5x)\cos(3x)\,\mathrm{d}x$$



Section 8.4

Trigonometric Substitutions

Monday, February 21, 2022

Trigonometric Substitutions

Goal:

Use properties and identities of trigonometric functions to rewrite algebraic expressions involving $a^2 - x^2$, $a^2 + x^2$, and $x^2 - a^2$ into integrable trigonometric forms.

Integrals Involving $a^2 - x^2$

Relevant Identity: $1 - \sin^2\theta = \cos^2\theta$

Substitution: $x = a\sin\theta$

Restriction on Substitution: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Example 8.4.1

Evaluate each integral.

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$

$$\int \frac{4}{x^2\sqrt{16-x^2}}\,\mathrm{d}x$$

Integrals Involving $a^2 + x^2$

Relevant Identity: $1 + \tan^2\theta = \sec^2\theta$

Substitution: $x = a\tan\theta$

Restriction on Substitution: $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Example 8.4.2

Evaluate each integral.

$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}}$$

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,\mathrm{d}x$$

Integrals Involving $x^2 - a^2$

Relevant Identity: $\sec^2\theta - 1 = \tan^2\theta$

Substitution: $x = a\sec\theta$

Restriction on Substitution: $0 \le \theta < \frac{\pi}{2}$ if $x \ge a$

$\frac{\pi}{2} < \theta \le \pi$ if $x \le -a$

Example 8.4.3

Evaluate each integral.

$$\int \frac{dx}{\sqrt{x^2 - 1}}, \qquad x > 1$$

$$\int x^3 \sqrt{x^2 - 25}\,dx, \qquad x > 5$$

Example 8.4.4

Evaluate each integral.

$$\int (36 - 9x^2)^{-3/2} \mathrm{d}x$$

$$\int \frac{\mathrm{d}x}{x^2\sqrt{9x^2 - 1}}, \qquad x > \frac{1}{3}$$

Example 8.4.5

Evaluate the integral

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x.$$

Example 8.4.6

Cut a 14’’ pizza with two parallel cuts into three parts of equal area.

Section 8.5

Partial Fractions

Wednesday, February 23, 2022

Partial Fractions

The Method of Partial Fractions allows us to rewrite a proper rational expression as the sum (or difference) of two or more simpler rational expressions.

Goals:

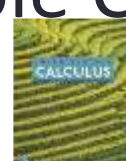
1. Discuss the Method of Partial Fractions
2. Use it to evaluate integrals.

Factoring Polynomials

Every polynomial with real coefficients can be uniquely factored into a product of linear and/or irreducible quadratic factors.

We need to consider four cases:

1. Distinct Linear Factors
2. Repeated Linear Factors
3. Distinct Irreducible Quadratic Factors
4. Repeated Irreducible Quadratic Factors



Case 1: Distinct Linear Factors

Consider the proper rational function $R(x) = \frac{P(x)}{Q(x)}$. If $Q(x)$ can be factored into distinct linear factors as $Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$, then the partial fraction decomposition of $R(x)$ is of the form

$$R(x) = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}$$

Example 8.5.1

Find the partial fraction decomposition of

$$\frac{4x+5}{x^2+x-2}.$$

Example 8.5.2

Evaluate each integral.

$$\int \frac{x + 14}{x^2 - 2x - 8}\,\mathrm{d}x$$

$$\int \frac{21x^2\,\mathrm{d}x}{x^3 - x^2 - 12x}$$

Case 2: Repeated Linear Factors

Consider the proper rational function $R(x) = \frac{P(x)}{Q(x)}$. If $Q(x)$ has a repeated linear factor $(ax + b)^n$ where $n \geq 2$ is an integer, then the partial fraction decomposition of $R(x)$ will include the terms

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n}$$

Example 8.5.3

Find the partial fraction decomposition of

$$\frac{x^2 - 3x + 5}{(x-2)^2(x+4)}.$$

Example 8.5.4

Evaluate the integral

$$\int_{-1}^{1} \frac{x}{(x+3)^2}\,\mathrm{d}x.$$

Case 3: Irreducible Quadratic Factors

Consider the proper rational function $R(x) = \frac{P(x)}{Q(x)}$. If $Q(x)$ has a distinct irreducible quadratic factor $(ax^2 + bx + c)$, then the partial fraction decomposition of $R(x)$ will include the term

$$\frac{Ax + B}{ax^2 + bx + c}$$

Example 8.5.5

Find the partial fraction decomposition of

$$\frac{x-3}{x^3+3x}.$$

Case 4: Repeated Quadratic Factors

Consider the proper rational function $R(x) = \frac{P(x)}{Q(x)}$. If $Q(x)$ has a repeated irreducible quadratic factor $(ax^2 + bx + c)^n$ where $n \geq 2$ is an integer, then the partial fraction decomposition of $R(x)$ will include the terms

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}$$

Example 8.5.6

Find the partial fraction decomposition of

$$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}.$$

Example 8.5.7

Evaluate each integral.

$$\int \frac{x^3 - 10x^2 + 27x}{x^2 - 10x + 25}\,\mathrm{d}x$$

$$\int \frac{e^{3x} + e^{2x} + e^x}{(e^{2x} + 1)^2}\,\mathrm{d}x$$

Summary

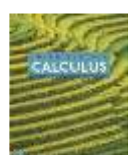
To form the partial fraction decomposition of a proper rational expression:

- Factor the denominator completely.
- Any distinct (non-repeated) factor produces one fraction.
- Any repeated factor creates a fraction with every power up to and including the multiplicity of the factor.
- The numerator of each fraction is of degree one less than the degree of the factor in the denominator (ignoring multiplicity).

Section 8.6

Integration Strategies

Monday, February 28, 2022



Which Technique Should We Use?

Two strategies are frequently recommended for determining the technique of integration to be used for a particular integral:

1. Hierarchical progression through the techniques from easiest to most complex.
2. Analyzing the integrand to determine the most likely technique to succeed.

In practice, a combination of the two strategies is best.

Which Techniques Do We Know?

- Elementary anti-differentiation (power rule, simple trig, exponential, …)
- u-substitution
- Integration by Parts
- Trigonometric Integration
  - Really, this is just u-sub!
- Trigonometric Substitution
- Partial Fraction Decomposition

Example 8.6.1

Evaluate each integral.

$$\int \frac{2\cos x + \cot x}{1 + \sin x}\,\mathrm{d}x$$

$$\int \frac{x^{-2} + x^{-3}}{x^{-1} + 16x^{-3}}\,\mathrm{d}x$$

Example 8.6.2

Evaluate each integral.

$$\int \sin(2x)\ln(\sin x)\,\mathrm{d}x$$

$$\int \sin(x)\ln(\sin x)\,\mathrm{d}x$$

Example 8.6.3

Evaluate each integral.

$$\int x \sin^{-1} x \, \mathrm{d}x$$

$$\int_{1}^{2} w^3 e^{w^2} \mathrm{d}w$$

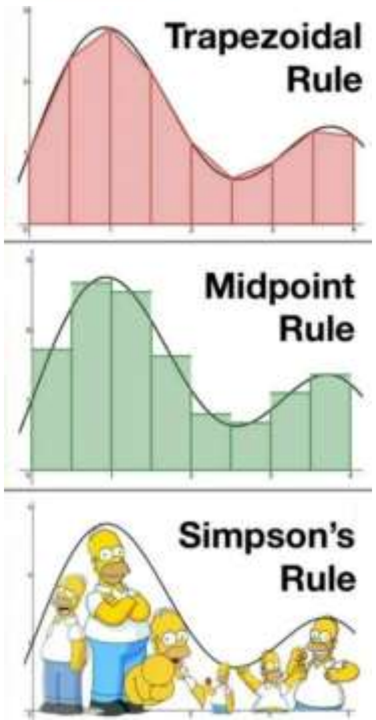
Example 8.6.4

Evaluate the integral

$$\int_{\frac{1}{e}}^{1} \frac{\mathrm{d}x}{x(\ln^2 x + 2\ln x + 2)}.$$

Read Your Textbook!

Your textbook goes through a number of interesting examples in this section. We *strongly* suggest you read this section in your textbook before attempting the Section 8.6 homework.



Section 8.8

Numerical Integration

Wednesday, March 2, 2022

Numerical Integration

When analytical methods of integration fail, we must turn to numerical methods. These methods do not produce exact values; instead, they produce approximations which are, in general, quite accurate.

These methods are often implemented in computer algebra systems and other software packages.

Numerical Integration

We will look at three methods:

- Midpoint Rule
- Trapezoidal Rule
- Simpson’s Rule

Absolute and Relative Error

If a computed numerical solution C arises from a problem having exact solution E, then the absolute error of the approximation is

$$|C - E|$$

and the relative error of the approximation is

$$\frac{|C - E|}{|E|}$$

(provided $E \neq 0$).

Calculus I: Riemann Sums

If the interval $[a, b]$ is partitioned into n subintervals of equal width $\Delta x = \frac{b-a}{n}$, then a Riemann sum can be used to approximate the value of $\int_a^b f(x)dx$.

Specifically,

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

where x_i is any point in the ith subinterval of $[a, b]$.

The Midpoint Rule

Suppose f is defined and integrable on $[a, b]$.

The Midpoint Rule approximation to $\int_a^b f(x)\mathrm{d}x$ using n subintervals of equal width $\Delta x = \frac{b-a}{n}$ is

$$M(n) = \big(f(m_1) + f(m_2) + \ldots + f(m_n)\big)\Delta x$$

where $m_i = \frac{x_{i-1}+x_i}{2}$ is the midpoint of the ith subinterval $[x_{i-1}, x_i]$ of $[a, b]$ and $x_i = a + i\Delta x$.

This is simply a midpoint Riemann sum.

Example 8.8.1

Find the Midpoint Rule approximations of the integral

$$\int_{1}^{9} x^3 \mathrm{d}x$$

using first $n = 4$ and then $n = 8$ subintervals.

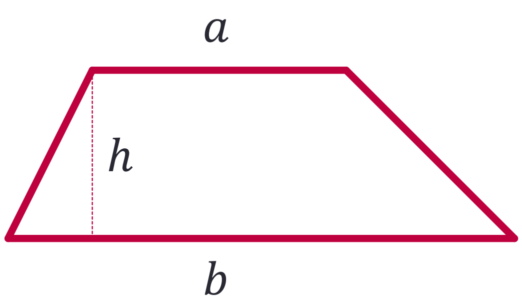
Calculate the absolute and relative error of each approximation. Round to three decimal places.

Are Rectangles the Best Option?

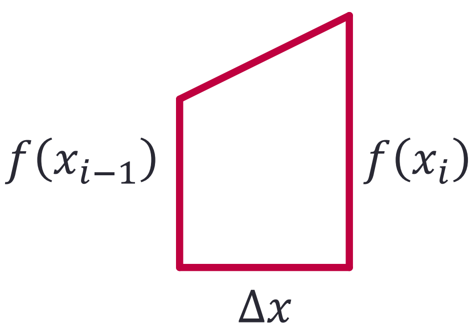
The Midpoint Rule makes the assumption that the function f consists of horizontal line segments on each subinterval, forming rectangles.

Another option: Assume that the function f consists of line segments connecting the endpoints of each subinterval, forming trapezoids.

The Area of a Trapezoid



$$A = \left(\frac{a+b}{2}\right)h$$



$$A = \left(\frac{f(x_{i-1}) + f(x_i)}{2}\right)\Delta x$$

The Trapezoid Rule

Suppose f is defined and integrable on $[a, b]$. The Trapezoid Rule approximation to $\int_a^b f(x)\mathrm{d}x$ using n subintervals of equal width $\Delta x = \frac{b-a}{n}$ is

$$T(n) = \left(\frac{f(x_0)}{2} + f(x_1) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right)\Delta x$$

where $x_i = a + i\Delta x$.

Example 8.8.2

Find the Trapezoid Rule approximations of the integral

$$\int_{1}^{9} x^{3}\,\mathrm{d}x$$

using first $n = 4$ and then $n = 8$ subintervals.

Calculate the absolute and relative error of each approximation. Round to three decimal places.

Comparing Midpoint and Trapezoid

The absolute error bounds associated with the Midpoint and Trapezoid Rules are both proportional to $(\Delta x)^2$.

Thus, each time Δx is reduced by a factor of 2 (or the number of subintervals doubles), the errors roughly decrease by a factor of 4.

Comparing Midpoint and Trapezoid

Both Midpoint and Trapezoid are more accurate than the endpoint approximations we focused on in Calculus I.

In general, for a given n (or Δx), the size of the error associated with the Midpoint Rule is about half the size of the error associated with the Trapezoid Rule.

Example 8.8.3

Suppose the points $(-h, y_0)$, $(0, y_1)$, and (h, y_2) lie on the parabola $p(x) = Ax^2 + Bx + C$. Compute the integral

$$\int_{-h}^{h} p(x)\mathrm{d}x$$

and express the solution in terms of y_0, y_1, y_2 and h.

Example 8.8.4

Use the area under a parabola to approximate the area under the function $y = f(x)$ on the interval $[a, b]$ using 6 subintervals of equal width.

Simpson's Rule

Suppose f is defined and integrable on $[a, b]$, and let $n \geq 2$ be an even integer. Simpson's Rule approximation $S(n)$ to $\int_a^b f(x)\mathrm{d}x$ using n subintervals of equal width $\Delta x = \frac{b-a}{n}$ is

$$S(n) = \left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + \cdots + 4f(x_{n-1}) + f(x_n)\right)\frac{\Delta x}{3}$$

where $x_i = a + i\Delta x$.

Example 8.8.5

Find the Simpson Rule approximations of the integral

$$\int_{1}^{9} x^3 \mathrm{d}x$$

using $n = 4$ subintervals.

Calculate the absolute and relative error of the approximation. Round to three decimal places.

Comparing Midpoint, Trapezoid, and Simpson's

The absolute error bounds associated with the Midpoint and Trapezoid Rules are both proportional to $(\Delta x)^2$.

The absolute error bounds associated with Simpson's Rule is proportional to $(\Delta x)^4$.

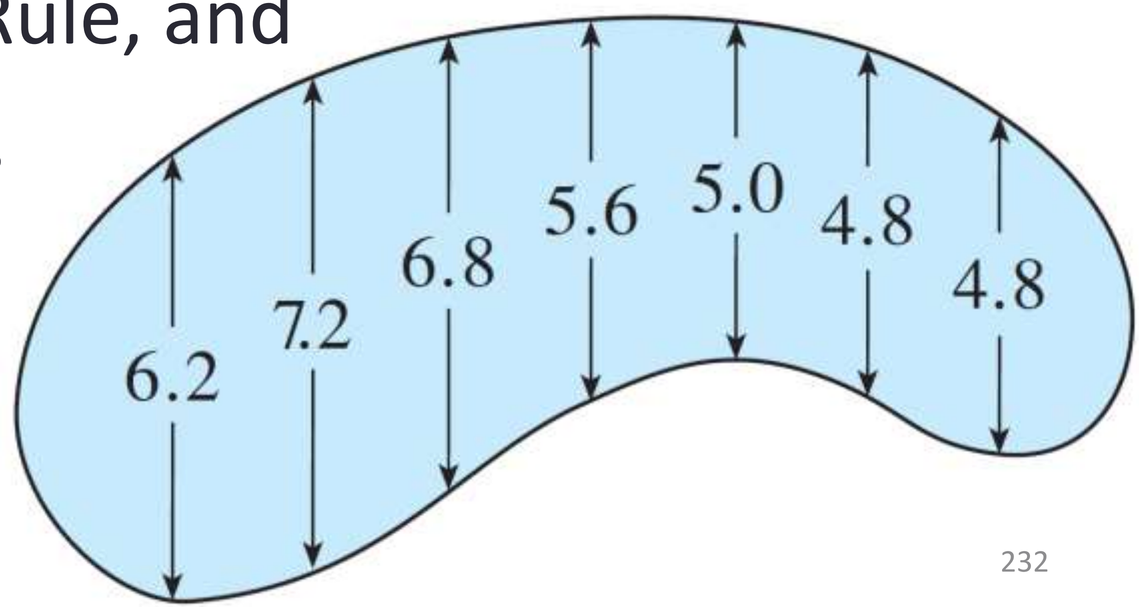
Thus, each time Δx is reduced by a factor of 2, the errors for Simpson's Rule roughly decrease by a factor of 16.

Example 8.8.6

The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Estimate the area of the pool (to 1 decimal place) using

a) The Trapezoid Rule, and

b) Simpson's Rule.



Section 8.9

Improper Integrals

Friday, March 4, 2022

Fundamental Theorem of Calculus (Part II)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$.

Improper Integrals

Improper Integrals occur when

- the interval of integration is infinite

and/or

- the integrand is discontinuous

Both of these cases can be handled easily using limits in conjunction with the Fundamental Theorem (Part II).

Improper Integrals over Infinite Intervals

If f is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\mathrm{d}x$$

provided the limit exists.

FTOC II applies to this integral!

Improper Integrals over Infinite Intervals

If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\mathrm{d}x = \lim_{a \to -\infty} \int_{a}^{b} f(x)\mathrm{d}x$$

provided the limit exists.

FTOC II applies to this integral!

Terminology

If the associated limit exists, we say that the improper integral converges.

If the associated limit does not exist, we say that the improper integral diverges.

Example 8.9.1

Evaluate the following integrals or state that they diverge.

$$\int_{1}^{\infty} \frac{\mathrm{d}x}{x}$$

$$\int_{1}^{\infty} \frac{\mathrm{dx}}{x^2}$$

Example 8.9.2

Evaluate the following integrals or state that they diverge.

$$\int_{-\infty}^{-2} \frac{1}{z^2} \sin\frac{\pi}{z}\, dz$$

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\, dx$$

Improper Integrals with Discontinuous Integrands

If $f(x)$ is continuous on the interval $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{c \to a^+} \int_c^b f(x)\mathrm{d}x$$

provided the limit exists.

FTOC II applies to this integral!

Improper Integrals with Discontinuous Integrands

If $f(x)$ is continuous on the interval $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{c \to b^-} \int_a^c f(x)\mathrm{d}x$$

provided the limit exists.

FTOC II applies to this integral!

Example 8.9.3

Evaluate the following integrals or state that they diverge.

$$\int_{-2}^{14} \frac{\mathrm{d}x}{\sqrt[4]{x+2}}$$

$$\int_{0}^{2} z^2 \ln z \,\mathrm{d}z$$

Comparison Theorem for Improper Integrals

Suppose the functions f and g are continuous on the interval $[a, \infty)$, and further suppose that $f(x) \geq g(x) \geq 0$ for $x \geq a$.

If $\int_a^\infty f(x)\mathrm{d}x$ converges, then $\int_a^\infty g(x)\mathrm{d}x$ converges.

If $\int_a^\infty g(x)\mathrm{d}x$ diverges, then $\int_a^\infty f(x)\mathrm{d}x$ diverges.

Example 8.9.4

Evaluate the following integrals or state that they diverge.

$$\int_{1}^{\infty} \frac{x+1}{\sqrt{x^4 - x}}\,\mathrm{d}x$$

$$\int_{0}^{1} \frac{e^{-x}}{\sqrt{x}}\,\mathrm{d}x$$

Section 10.2

Sequences

Friday, March 11, 2022

Note: Section 10.1 is partly included

Sequences

A sequence is an ordered list of numbers of the form $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$.

Alternate Notation:

$$\{a_n\}$$

$$\{a_n\}_{n=2}^{\infty}$$

We assume sequences begin with $n = 1$ unless explicitly directed otherwise.

Explicitly Defined Sequences

Some (but not all) sequences can be defined by giving an explicit formula for the n^{th} term of the sequence.

Example:

The formula

$$a_n = (-1)^n \left(\frac{n}{n+1}\right)$$

defines the sequence

$$\left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \dots\right\}$$

Example 10.2.1

Consider the sequence

$$\left\{\frac{2}{3}, \frac{5}{6}, \frac{8}{9}, \frac{11}{12}, \cdots\right\}$$

a) Find the next two terms in the sequence.

b) Write an expression for the n^{th} term of the sequence.

Example 10.2.2

Consider the sequence

$$\left\{1, \frac{3}{2}, \frac{9}{5}, 2, \frac{15}{7}, \frac{9}{4}, \dots\right\}$$

a) Find the next two terms in the sequence.

b) Write an expression for the n^{th} term of the sequence $\{a_n\}_{n=1}^{\infty}$.

Example 10.2.3

Consider the sequence

$$\left\{1, -x, \frac{x^2}{2}, -\frac{x^3}{6}, \frac{x^4}{24}, -\frac{x^5}{120}, \dots\right\}$$

a) Find the next two terms in the sequence.

b) Write an expression for the n^{th} term of the sequence.

Factorials

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

n factorial

Special Case:

$$0! = 1$$

Note that factorials are only defined for nonnegative integers.

Example 10.2.4

Simplify the following factorial expression.

$$\frac{(2n+1)!}{(2n-1)!}$$

Recursively Defined Sequences

Some (but not all) sequences can be defined by giving one (or more) initial terms and an equation which defines the n^{th} term in terms of one (or more) previous terms.

Example:

$$a_n = a_{n-1} - 2;\ a_1 = 5$$

defines the sequence

$$\{5, 3, 1, -1, -3, \ldots\}$$

Example 10.2.5: The Fibonacci Sequence

Each term of the Fibonacci sequence is the sum of the two preceding terms, with initial conditions $f_1 = f_2 = 1$.

a) Find the sixth term in the Fibonacci sequence.
b) Write the recursive definition of the Fibonacci sequence.

The Limit of a Sequence

If the terms of a sequence $\{a_n\}$ approach a finite value L as $n \to \infty$, then we say the limit

$$\lim_{n\to\infty} a_n = L$$

exists and the sequence converges to L.

If the terms of a sequence $\{a_n\}$ do not approach a finite value L as $n \to \infty$, then we say the sequence diverges.

The Limit of a Sequence

Suppose f is a function such that $f(n) = a_n$ for every relevant nonnegative integer n. Then,

$$\lim_{x \to \infty} f(x) = \lim_{n \to \infty} a_n$$

provided $\lim_{x \to \infty} f(x)$ exists or is infinite.

Example 10.2.6

Consider the sequence whose n^{th} term is

$$a_n = \frac{n^2}{2^n - 1}$$

a) Write the first four terms of the sequence.

b) Show that the sequence converges.

Terminology for Sequences

$\{a_n\}$ is increasing if $a_{n+1} > a_n$

$\{a_n\}$ is nondecreasing if $a_{n+1} \geq a_n$

$\{a_n\}$ is decreasing if $a_{n+1} < a_n$

$\{a_n\}$ is nonincreasing if $a_{n+1} \leq a_n$

$\{a_n\}$ is monotonic if it is either nonincreasing or nondecreasing.

Example 10.2.7

Determine whether the sequence is increasing, nondecreasing, decreasing, nonincreasing, or not monotonic.

$$a_n = \frac{n}{2^{n+2}}$$

Terminology for Sequences

$\{a_n\}$ is bounded above if there exists a number M such that $a_n \leq M$ for all relevant values of n.

$\{a_n\}$ is bounded below if there exists a number N such that $a_n \geq N$ for all relevant values of n.

$\{a_n\}$ is a bounded sequence if it is both bounded above and bounded below.

Example 10.2.7 (continued)

Determine whether the sequence is bounded.

$$a_n = \frac{n}{2^{n+2}}$$

Theorem

All bounded monotonic sequences converge.

Thus, the sequence

$$a_n = \frac{n}{2^{n+2}}$$

from Example 10.2.7 is convergent!

Example 10.2.8

Determine whether the sequence whose n^{th} term is

$$a_n = \frac{4^n}{7^{n+1}}$$

converges or diverges.

Geometric Sequences

Geometric sequences have the form $\{r^n\}$ or $\{ar^n\}$. Each term is obtained by multiplying the previous term by a fixed constant.

Sequences of the form $\{r^n\}$ are

- Convergent to zero for $|r| < 1$
- Convergent to one for $r = 1$
- Divergent for $r > 1$ and for $r \leq -1$

Example 10.2.9

Find the limit of each sequence or determine that the sequence diverges.

$$\left\{4\left(\frac{2}{3}\right)^n\right\}$$

$$\{7(-1.001)^n\}$$

Example 10.2.10

Consider the sequence whose n^{th} term is

$$a_n = \frac{\cos(n\pi)}{n^2}$$

a) Write the first four terms of the sequence.

b) Determine the convergence or divergence of the sequence.

Σ

Section 10.3

Series

Monday, March 14, 2022

Note: Section 10.1 is partly included

Series

Given a sequence $\{a_1, a_2, a_3, \ldots\}$, the sum of its terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

is called an infinite series.

Partial Sums

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

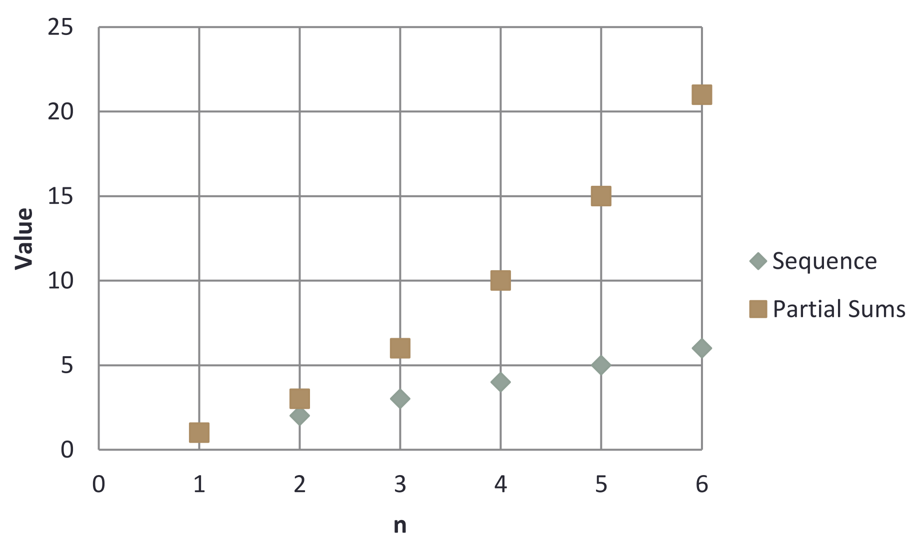
The sequence of partial sums $\{S_n\}$ is used to analyze the behavior of the series.

Example 10.3.1

Consider the sequence $a_n = n$.

a) Find the first six terms of the sequence.

b) Find the first six partial sums of the series $\sum_{n=1}^{\infty} a_n$.

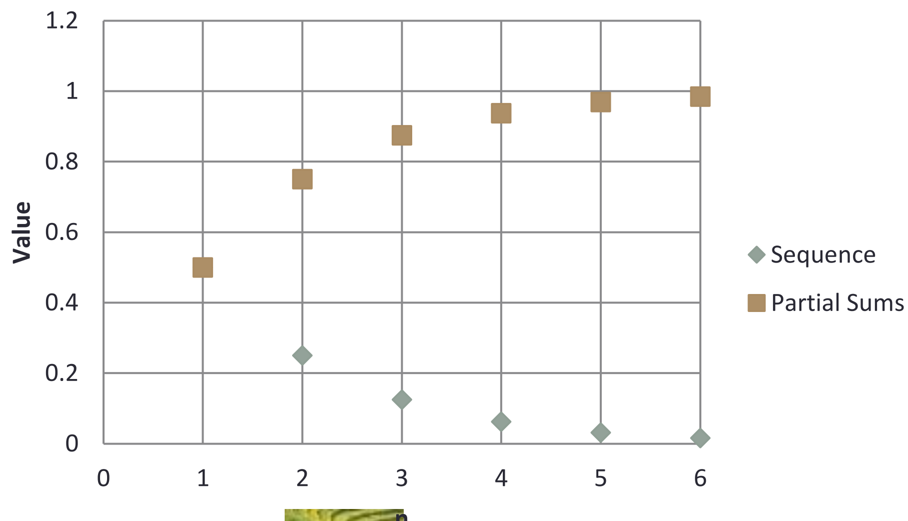


Example 10.3.2

Consider the sequence $a_n = \dfrac{1}{2^n}$.

a) Find the first six terms of the sequence.

b) Find the first six partial sums of the series $\sum_{n=1}^{\infty} a_n$.



Convergence of Series

If the sequence of partial sums $\{S_n\}$ converges to a finite value L, then the associated series converges to L and we write

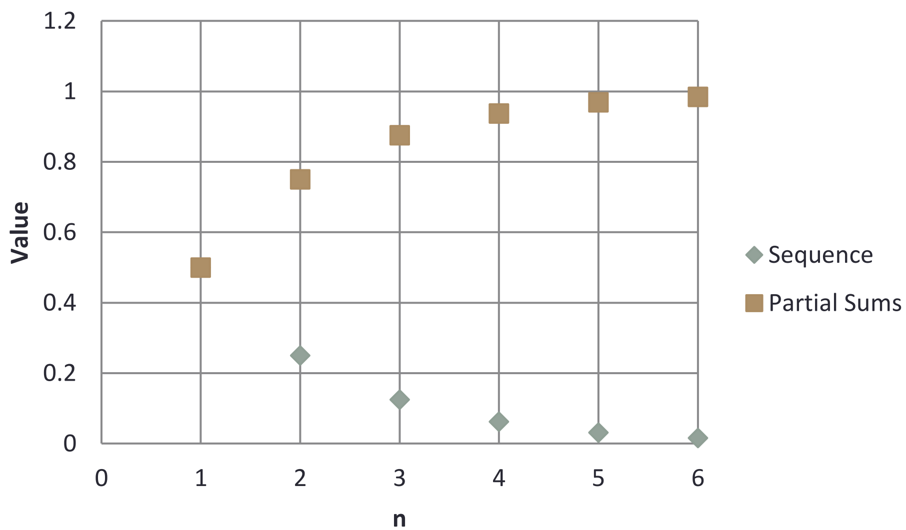
$$\sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums $\{S_n\}$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 10.3.2 (continued)

Consider the sequence $a_n = \frac{1}{2^n}$.

c) Determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.



Recall: Geometric Sequences

Geometric sequences have the form $\{r^n\}$ or $\{ar^n\}$. Each term is obtained by multiplying the previous term by a fixed constant.

Sequences of the form $\{r^n\}$ are

- Convergent to zero for $|r| < 1$
- Convergent to one for $r = 1$
- Divergent for $r > 1$ and for $r \leq -1$

Geometric Series

The series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

where $a \neq 0$ and r is a real number is called a geometric series with ratio r.

Note: Many sources express this series as

$$\sum_{n=1}^{\infty} ar^{n-1}$$

Geometric Sums

The partial sums of a geometric series are called geometric sums.

A geometric sum with k terms has the form

$$S_k = \sum_{n=0}^{k-1} ar^n = a + ar + ar^2 + \cdots + ar^{k-1}$$

where $a \neq 0$ and r is a real number.

Example 10.3.3

Find a formula for the value of the geometric sum

$$S_k = \sum_{n=0}^{k-1} ar^n = a + ar + ar^2 + \cdots + ar^{k-1}$$

where $|r| < 1$.

Then, evaluate $\lim_{k\to\infty} S_k$.

Geometric Series

If $|r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ converges. Specifically,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

If $|r| \geq 1$, then the geometric series diverges.

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$(r \neq 1)$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$(|r| < 1)$

Example 10.3.4

Determine whether each series is convergent or divergent. If it is convergent, find its sum.

$$8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$$

$$\sum_{n=0}^{\infty} 5^n 4^{1-n}$$

Example 10.3.5

Write each repeating decimal as the ratio of two integers by first writing it as a geometric series.

$$0.\overline{9}$$

$$1.\overline{03}$$

Example 10.3.6

Find the sum of the convergent series.

$$\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)}$$

Properties of Convergent Series

Let $\sum a_n$ converge to A, let $\sum b_n$ converge to B, and let c be a real number. Then, the following series are also convergent.

$$\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$$

$$\sum (ca_n) = c \sum a_n = cA$$

Properties of Divergent Series

Let $\sum a_n$ diverge, let $\sum b_n$ converge to B, and let c be a nonzero real number. Then, the following series are divergent.

$$\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$$

$$\sum (c a_n) = c \sum a_n$$

Properties of Series

If M is a positive integer, then

$$\sum_{n=1}^{\infty} a_n$$

and

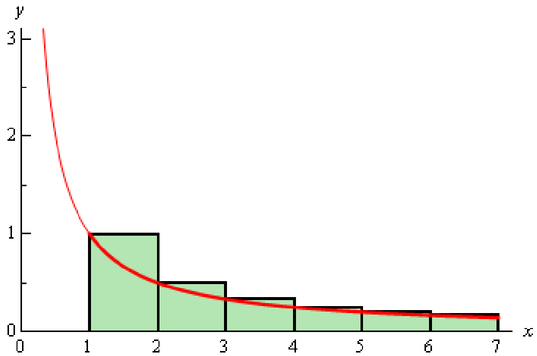
$$\sum_{n=M}^{\infty} a_n$$

either both converge or both diverge.

Example 10.3.7

Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{k=1}^{\infty}\left(\left(\frac{1}{6}\right)^{k}+\left(\frac{1}{3}\right)^{k-1}\right)$$



Section 10.4

The Divergence and Integral Tests

Monday, March 21, 2022

The Divergence Test

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

If $\lim_{n\to\infty} a_n \neq 0$ or if the sequence $\{a_n\}$ does not converge, then the series $\sum a_n$ is divergent.

WARNING!

The convergence to zero of the sequence $\{a_n\}$ is not sufficient to demonstrate convergence of the series $\sum a_n$.

Example 10.4.1

Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$$

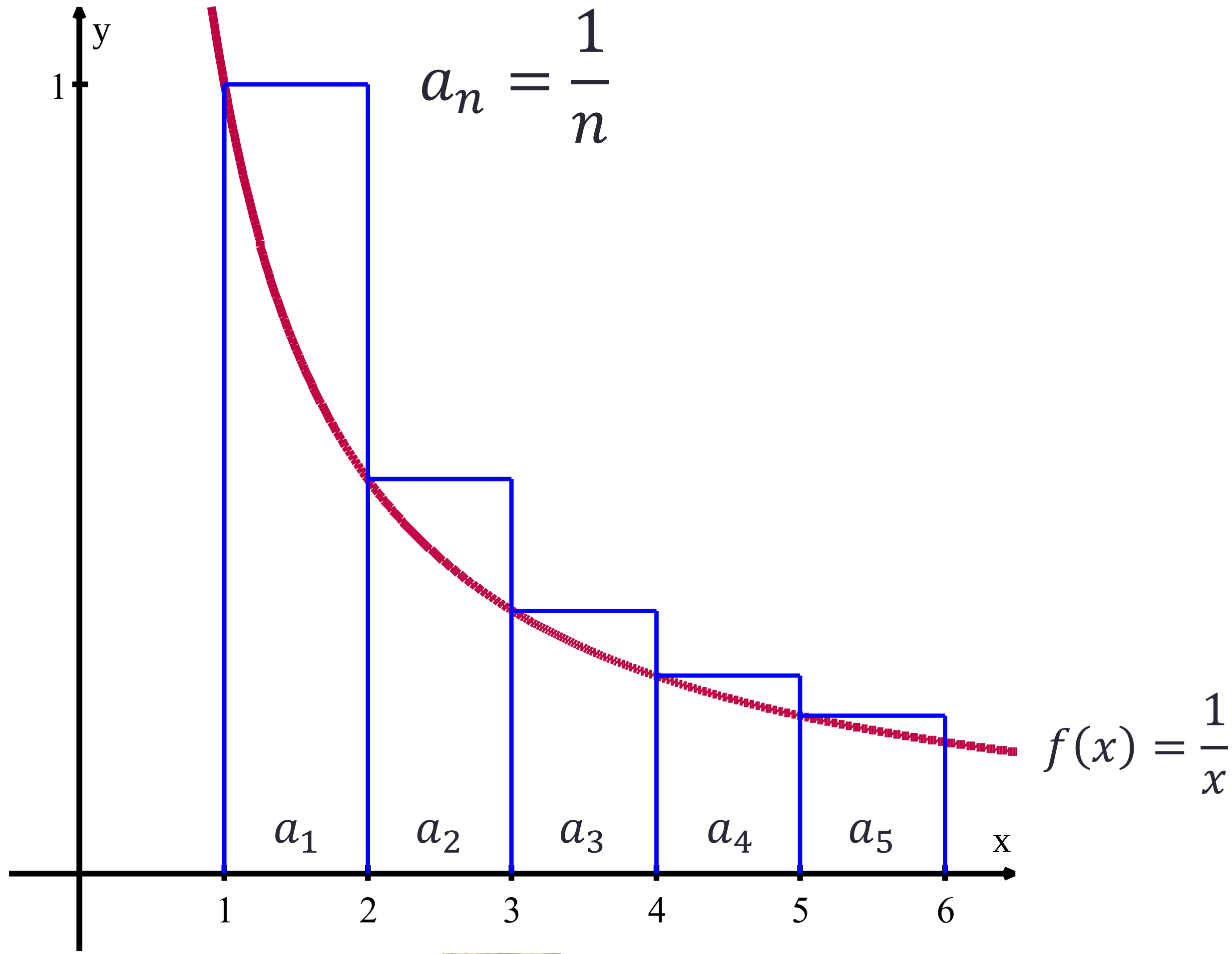
Example 10.4.2 – The Harmonic Series

Determine whether the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

converges or diverges.

Example 10.4.2 – The Harmonic Series



Example 10.4.2 – The Harmonic Series

On the previous slide, we demonstrated that

$$S_5 > \int_1^6 \frac{1}{x}\,\mathrm{d}x$$

In general, it is easy to see that

$$S_n > \int_1^{n+1} \frac{1}{x}\,\mathrm{d}x$$

Since the associated improper integral is divergent, the sequence of partial sums $\{S_n\}$ is also divergent.

The Harmonic Series

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges even though the terms of the associated sequence approach zero.

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$ for integers $n \geq 1$. Then, the series

$$\sum_{n=1}^{\infty} a_n$$

and the improper integral

$$\int_1^{\infty} f(x) \mathrm{d}x$$

either both converge or both diverge.

The Integral Test

In the case when both the series and the integral converge, the value of the integral is in general not equal to the value of the series.

Partial Proof of the Integral Test

Show that if the improper integral

$$\int_{1}^{\infty} f(x)\mathrm{d}x$$

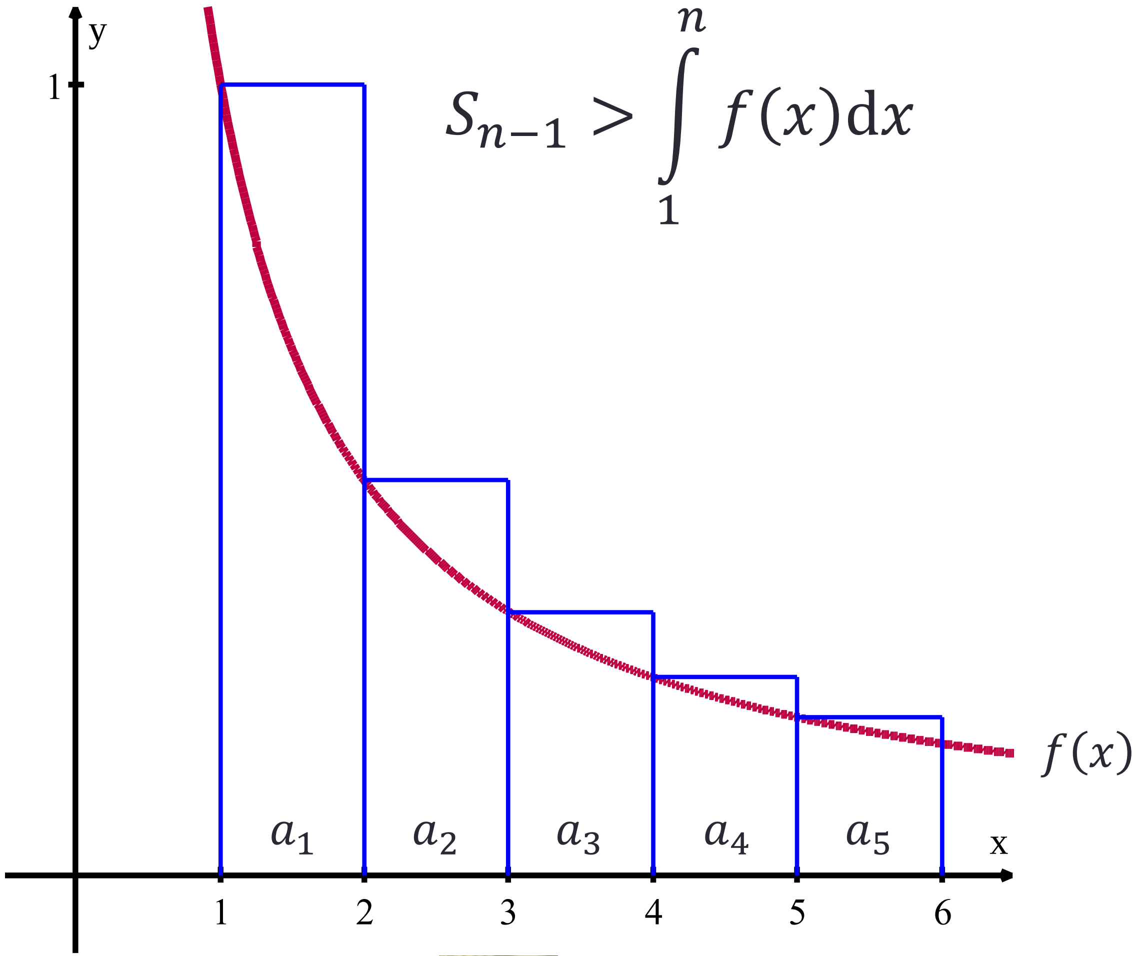
is divergent, then the infinite series

$$\sum_{n=1}^{\infty} a_n$$

must also be divergent.

Partial Proof of the Integral Test

$$S_{n-1} > \int_1^n f(x)\mathrm{d}x$$



Partial Proof of the Integral Test

Assume $\int_1^{\infty} f(x)\mathrm{d}x$ is divergent.

Then,

$$\int_1^{\infty} f(x)\mathrm{d}x = \lim_{n\to\infty} \int_1^{n} f(x)\mathrm{d}x$$
$$\leq \lim_{n\to\infty} (S_{n-1})$$

Thus, since the limit of the partial sums is greater than or equal to a divergent limit, it must also be divergent.

Example 10.4.3

Use the Integral Test to determine whether each series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$$

p-Series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a p-series.

The p-series is

- Convergent if $p > 1$
- Divergent if $p \leq 1$

Example 10.4.4

Determine the convergence or divergence of the series

$$\sum_{n=2}^{\infty} \frac{n^e}{n^\pi}.$$

Partial Sums as Estimates

Let $S = \sum_{k=1}^{\infty} a_k$.

To estimate the sum S of a series , we can use any partial sum S_n of the series.

The error (or the remainder) of the approximation S_n is

$$R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Estimating the Remainder

Let f be a continuous, positive, decreasing function for $x \geq 1$ and let $a_k = f(k)$ for positive integer values k. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series where the nth partial sum is $S_n = \sum_{k=1}^{n} a_k$.

Then, if $R_n = S - S_n$, it can be shown that

$$R_n \leq \int_{n}^{\infty} f(x)\mathrm{d}x$$

Example 10.4.5

Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

a) Determine the convergence of the series.

b) Use the sum of the first 10 terms to estimate the sum of the series.

c) Estimate the error involved in the approximation.

d) How many terms are required to ensure that the sum is accurate to within 0.001.

Estimating the Exact Value

Let f be a continuous, positive, decreasing function for $x \geq 1$ and let $a_k = f(k)$ for positive integer values k. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series where the nth partial sum is $S_n = \sum_{k=1}^{n} a_k$.

Then, the exact value of the series is bounded by

$$L_n = S_n + \int_{n+1}^{\infty} f(x)dx < \sum_{k=1}^{\infty} a_k < S_n + \int_{n}^{\infty} f(x)dx = U_n$$

Example 10.4.5 (continued)

Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

e) Find lower and upper bounds on the exact value of the series.

f) Find an interval in which the value of the series must lie if you approximate it using ten terms of the series.

Section 10.5

Comparison Tests

Friday, March 25, 2022

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $0 < a_n \leq b_n$ for all relevant n and if $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.

2. If $0 < b_n \leq a_n$ for all relevant n and if $\sum b_n$ is divergent, then $\sum a_n$ is also divergent.

Example 10.5.1

Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$

Example 10.5.2

Determine whether each series is convergent or divergent.

$$\sum_{k=1}^{\infty} \frac{1}{3^k + 2}$$

$$\sum_{k=1}^{\infty} \frac{1}{3^k - 2}$$

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms, and let

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

1. If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

2. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example 10.5.2 (continued)

Determine whether the series is convergent or divergent.

$$\sum_{k=1}^{\infty} \frac{1}{3^k - 2}$$

Example 10.5.3

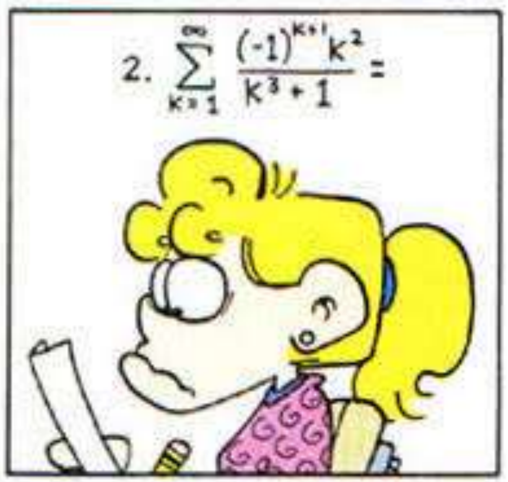
Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

Example 10.5.4

Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^4+n}}$$



Section 10.6

Alternating Series

Monday, April 4, 2022

Alternating Series

An alternating series is a series whose terms are alternately positive and negative.

Symbolically, an alternating series is of the form $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$ where $a_n > 0$ for all relevant n.

Example 10.6.1

Calculate the first four partial sums of each series.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^3}$$

Alternating Series Test

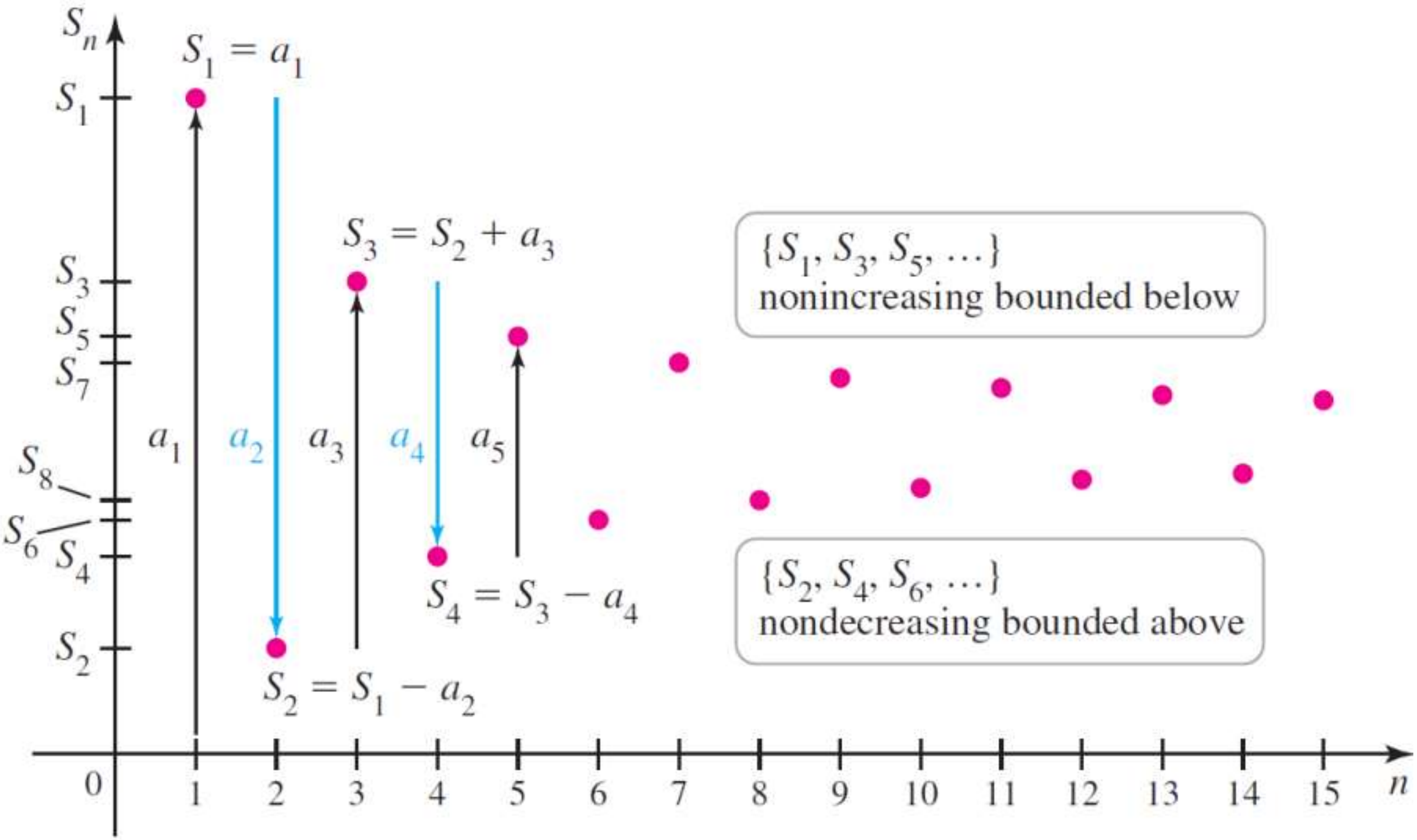
The alternating series

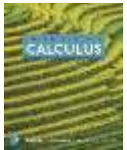
$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k , \qquad a_k > 0$$

converges provided

1. The terms of the series are nonincreasing in magnitude; that is, $a_{k+1} \leq a_k$ for all k greater than some positive integer N, and
2. $\lim_{k \to \infty} a_k = 0$

Alternating Series Test: Proof





Example 10.6.2

Determine whether the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

converges or diverges.

Caution About Convergence

For a series $\sum a_n$ of positive terms, recall that $\lim_{n\to\infty} a_n = 0$ does not imply convergence of the series.

For an alternating series $\sum (-1)^{n+1} a_n$ with terms which are nonincreasing in magnitude, $\lim_{n\to\infty} a_n = 0$ does imply convergence of the series.

Example 10.6.3

Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^3}$$

$$\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$$

Remainders in Alternating Series

If S is the sum of the convergent alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k$$

then the remainder $R_n = S - S_n$ satisfies

$$|R_n| \leq a_{n+1}$$

In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Example 10.6.4

Determine how many terms of the convergent series must be summed to be sure that the remainder is less than 10^{-4} in magnitude.

$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{8^k}$$

Absolute and Conditional Convergence

A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

Example 10.6.5

The alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

is convergent.

Is it absolutely or conditionally convergent?

Absolute Convergence Implies Convergence

If $\sum |a_n|$ is convergent, then $\sum a_n$ is convergent.

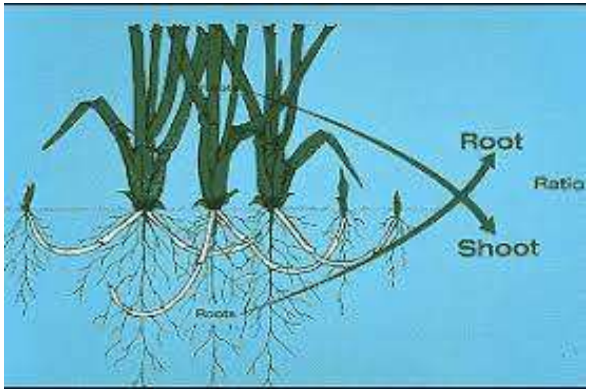
If $\sum a_n$ is divergent, then $\sum |a_n|$ is divergent.

Example 10.6.6

Determine whether each series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$$

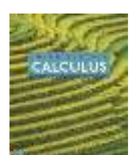
$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{4}\right)}{n^2}$$



Section 10.7

The Ratio and Root Tests

Wednesday, April 6, 2022



The Ratio Test

Consider the series $\sum a_n$ where $a_n \neq 0$.

1. If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ converges absolutely (and therefore converges).

2. If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| > 1$ or if $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \infty$, then $\sum a_n$ diverges.

3. If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive.

Example 10.7.1

Determine whether each series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{n!}$$

The Root Test

Consider the series $\sum a_n$.

1. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum a_n$ converges absolutely (and therefore converges).

2. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then $\sum a_n$ diverges.

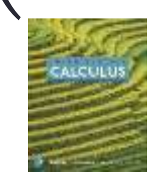
3. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the test is inconclusive.

Example 10.7.2

Determine whether each series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$$

$$\sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1}\right)^n$$



Example 10.7.3

A sequence is defined recursively by the equations

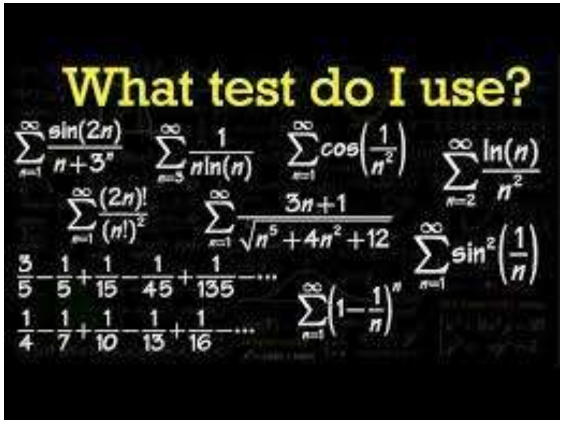
$$a_1 = 2 \qquad\qquad a_{n+1} = \frac{5n+1}{4n+3}a_n$$

Determine whether the associated series $\sum a_n$ is absolutely convergent, conditionally convergent, or divergent.

Example 10.7.4

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$$



Section 10.8

Choosing a Convergence Test

Monday, April 11, 2022

Which Test Should We Use?

Two strategies are frequently recommended for determining the convergence test to be used for a particular series:

1. Hierarchical progression through the tests from easiest to most complex.
2. Analyzing the series to determine the most likely technique to succeed.

In practice, a combination of the two strategies is best.

Which Tests Do We Know?

- *p*-series
- Geometric Series
- Divergence Test
- Alternating Series Test
- Integral Test
- Comparison Test
- Limit Comparison Test
- Ratio Test
- Root Test

Example 10.8.1

Determine whether each series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$$

Example 10.8.2

Determine whether each series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

$$\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$$

Example 10.8.3

Determine whether each series converges or diverges.

$$\sum_{k=1}^{\infty} k^2 e^{-k}$$

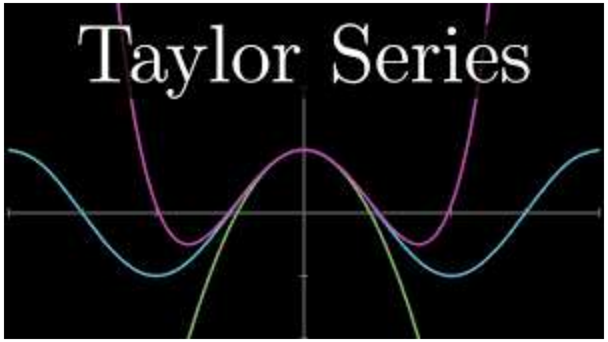
$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

Example 10.8.4

Determine whether each series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$$

$$\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$$



Section 11.1

Approximating Functions with Polynomials

Wednesday, April 13, 2022

Power Series

A power series is a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots$$

or, more generally,

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

where the center of the series a and the coefficients c_k are constants and x is a variable.

Partial sums of power series are polynomials!

Calculus I: Linear Approximations

Near the point of tangency, the tangent line to $y = f(x)$ provides a good approximation of the original function.

The linear approximation to $f(x)$ centered at a is

$$p_1(x) = f(a) + f'(a)(x - a)$$

Note that

$$p_1(a) = f(a)$$
$$p_1{}'(a) = f'(a)$$

Quadratic Approximations

The quadratic approximation to $f(x)$ centered at a is

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Note that

$$p_2(a) = f(a)$$
$$p_2{}'(a) = f'(a)$$
$$p_2{}''(a) = f''(a)$$

Example 11.1.1

Find the linear and quadratic approximating polynomials for

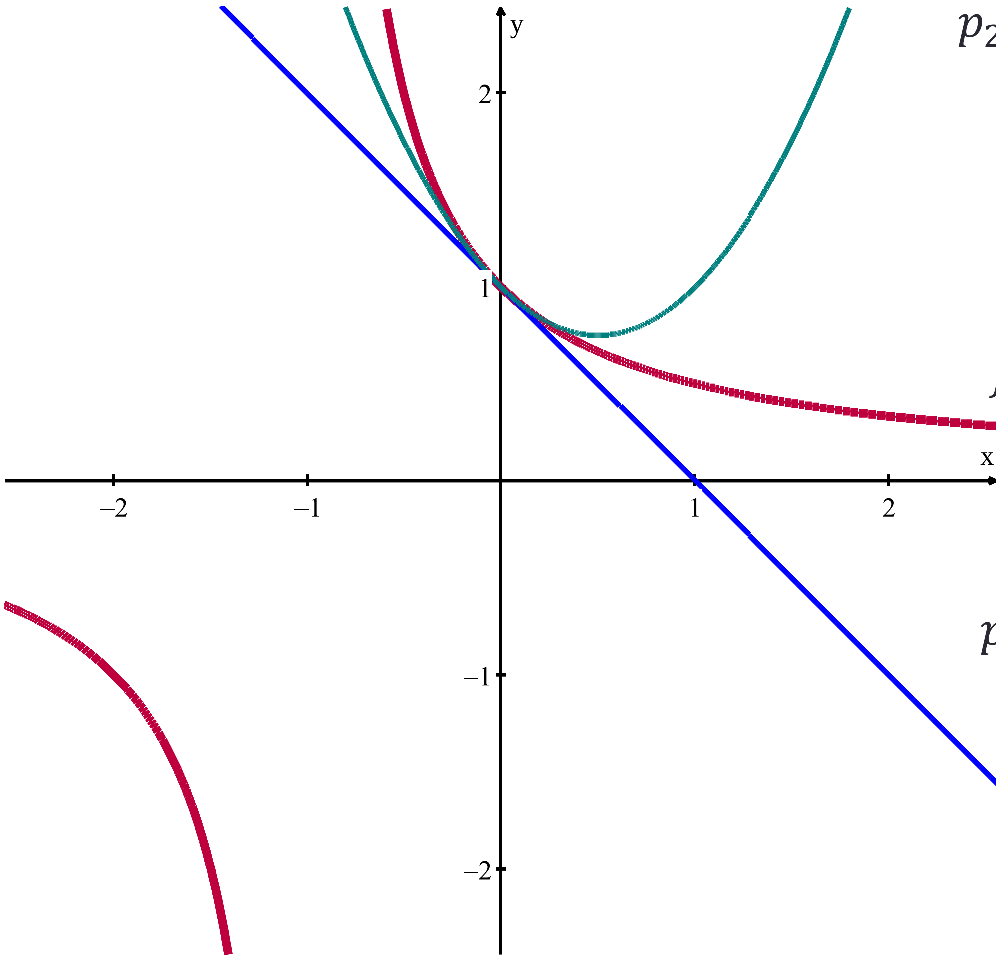
$$f(x) = \frac{1}{1+x}$$

centered at $a = 0$.

Use these polynomials to approximate $\frac{1}{1.05}$.

Graph the function and both polynomials.

Example 11.1.1



$p_2(x) = 1 - x + x^2$

$f(x) = \dfrac{1}{1+x}$

$p_1(x) = 1 - x$

Polynomial Approximation

If we wish to approximate a function $f(x)$ near $x = a$ using a polynomial of degree n, we need that approximating polynomial to equal $f(x)$ at the center point a.

Furthermore, we need the first n derivatives of the approximating polynomial and the function to agree at a (which means that those derivatives must all exist).

Taylor Polynomials

Let f be a function with $f, f', f'', \ldots,$ and $f^{(n)}$ defined at a. The nth-order Taylor polynomial for f centered at a is

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

p_n matches f in value, slope, and all derivatives up to the nth derivative at a.

Example 11.1.1 (revisited)

Find the Taylor polynomials p_3 and p_4 for

$$f(x) = \frac{1}{1+x}$$

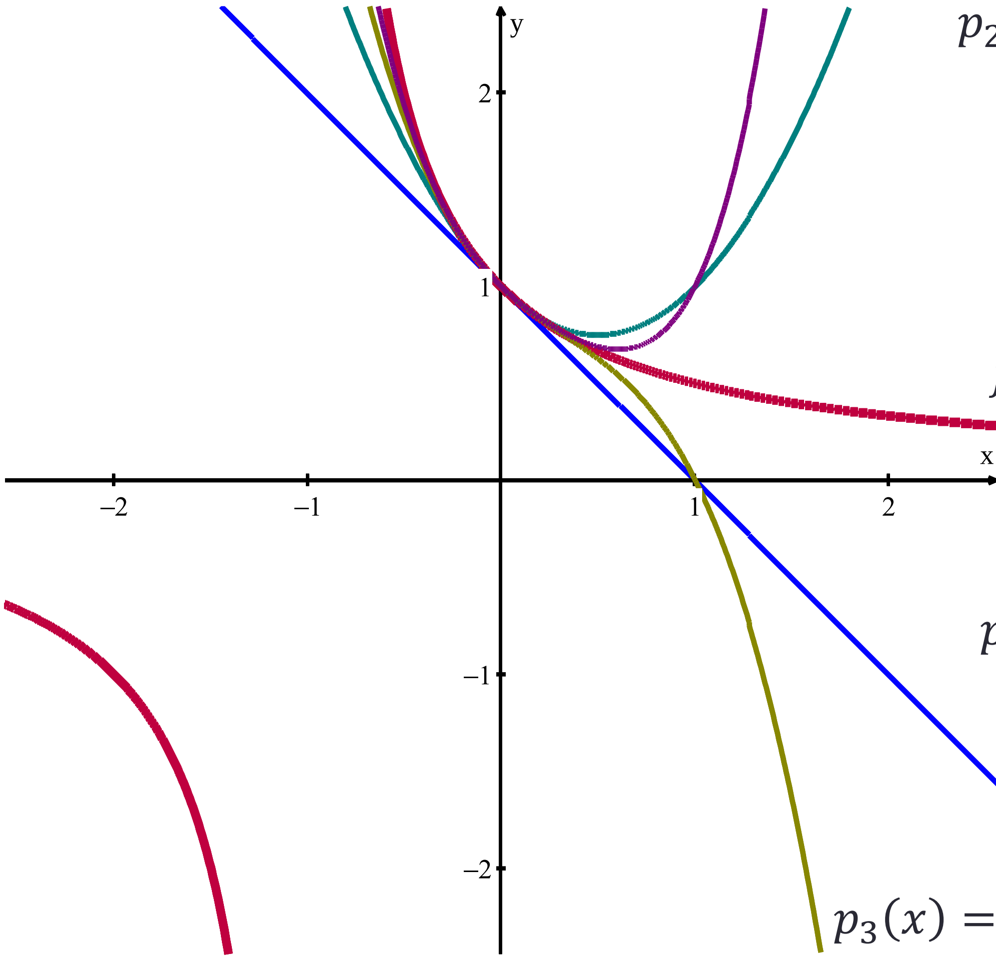
centered at $a = 0$.

Graph the function and both polynomials.

Example 11.1.1

$$p_4(x) = 1 - x + x^2 - x^3 + x^4$$

$$p_2(x) = 1 - x + x^2$$



$$f(x) = \frac{1}{1+x}$$

$$p_1(x) = 1 - x$$

$$p_3(x) = 1 - x + x^2 - x^3$$

Example 11.1.2

Find the Taylor polynomials p_1, p_2, and p_3 centered at $a = 1$ for $f(x) = x^3$.

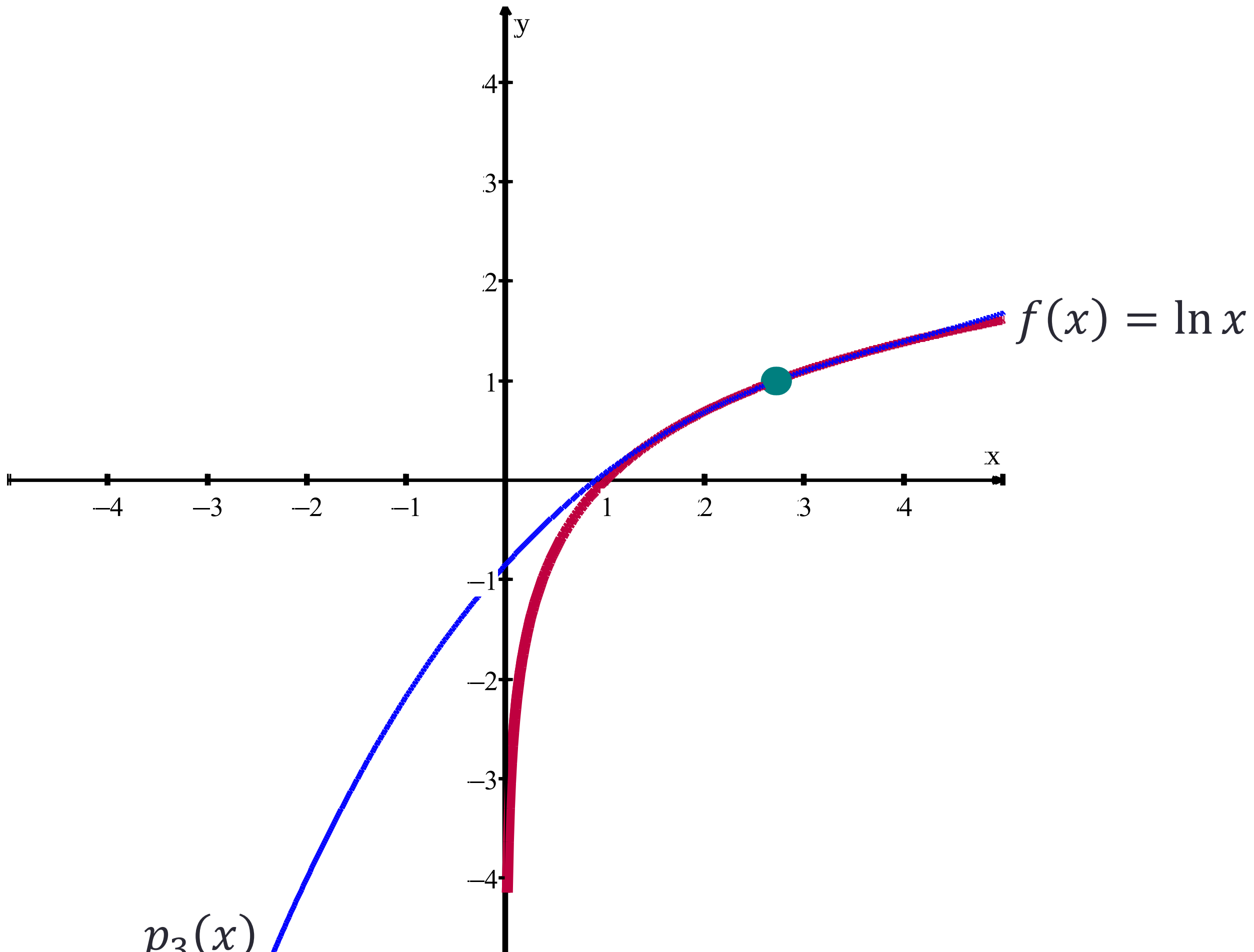
Example 11.1.3

Find the Taylor polynomial p_3 centered at $a = e$ for $f(x) = \ln x$.

Use p_3 to approximate $\ln 3$

Graph f and p_3.

Example 11.1.3



The Remainder in a Taylor Polynomial

Let p_n be the nth-order Taylor polynomial for f.

The remainder in using p_n to approximate f at the point x is

$$R_n(x) = f(x) - p_n(x)$$

The absolute value of the remainder is the error.

Example 11.1.4

Approximate $\sqrt{1.06}$ using a Taylor polynomial of degree 3.

Compute the absolute error in the approximation, assuming the exact value is given by a calculator.

Taylor's Theorem

Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. Then, for all x in I, we can express f as

$$f(x) = p_n(x) + R_n(x)$$

where p_n is the nth-order Taylor polynomial for f centered at a and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some point c between x and a.

Notes about Taylor's Theorem

The $(n+1)$st degree term in p_{n+1} is

$$\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}$$

The remainder R_n for p_n is

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is a point between x and a.

Example 11.1.5

Find the remainder R_n for the nth-order Taylor polynomial centered at $a = \dfrac{\pi}{2}$ for the function $f(x) = \cos x$. Express the result for a general value of n.

Find an upper and lower bound for the value of R_n associated with a general value of n.

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

Section 11.2

Properties of Power Series

Friday, April 15, 2022

Power Series

A power series has the general form

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where a and c_k are real numbers and x is a variable.

Power Series

$$\sum_{k=0}^{\infty} c_k(x-a)^k$$

The c_k's are the coefficients of the power series.

a is the center of the power series.

The set of values of x for which the series converges is its interval of convergence.

The distance from the center a to the endpoints of the interval of convergence is the radius of convergence R.

Convergence of Power Series

A power series

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

converges in one of three ways:

1) The series converges absolutely (and thus converges) for all x.
   The interval of convergence is $(-\infty, \infty)$.
   The radius of convergence is $R = \infty$.

Convergence of Power Series

A power series

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

converges in one of three ways:

2) There is a real number $R > 0$ such that the series converges absolutely (and thus converges) for $|x-a| < R$ and diverges for $|x-a| > R$. The radius of convergence is R.

Convergence of Power Series

A power series

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

converges in one of three ways:

3) The series converges only when $x = a$.
The radius of convergence is $R = 0$.

Example 11.2.1

Find the radius and interval of convergence for the power series.

$$\sum_{k=0}^{\infty} 3(x-2)^k$$

Example 11.2.2

Find the radius and interval of convergence for the power series.

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Example 11.2.3

Find the radius and interval of convergence for the power series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Recall: Geometric Series

If $|r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ converges. Specifically,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Geometric Power Series

If $|x| < 1$, the geometric power series $\sum_{n=0}^{\infty} x^n$ converges. Specifically,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

The radius of convergence is $R = 1$.

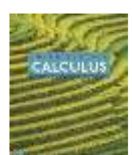
The interval of convergence is $(-1,1)$.

Combining Power Series

Suppose the power series $\sum c_k x^k$ converges to $f(x)$ and $\sum d_k x^k$ converges to $g(x)$ on an interval I.

$\sum (c_k \pm d_k) x^k$ converges to $f(x) \pm g(x)$ on I.

$x^m \sum c_k x^k = \sum c_k x^{k+m}$ converges to $x^m f(x)$ provided $x \neq 0$, m is an integer, and $k + m > 0$ for all terms of the series.



Combining Power Series

Suppose the power series $\sum c_k x^k$ converges to $f(x)$ and $\sum d_k x^k$ converges to $g(x)$ on an interval I.

If $h(x) = bx^m$ where m is a positive integer and b is a nonzero real number, then $\sum c_k \left(h(x)\right)^k$ converges to $f\left(h(x)\right)$.

Example 11.2.4

Find a power series representation for each function. Identify the interval of convergence.

$$f(x) = \frac{3}{1 - x^4}$$

$$g(x) = \frac{1}{x + 10}$$

Differentiating Power Series

Suppose the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ converges for $|x-a|<R$ and defines a function f on that interval. Then, f is differentiable for $|x-a|<R$; specifically,

$$f'(x)=\sum_{k=0}^{\infty} kc_k(x-a)^{k-1}$$

Integrating Power Series

Suppose the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ converges for $|x-a|<R$ and defines a function f on that interval. Then, f is integrable for $|x-a|<R$; specifically,

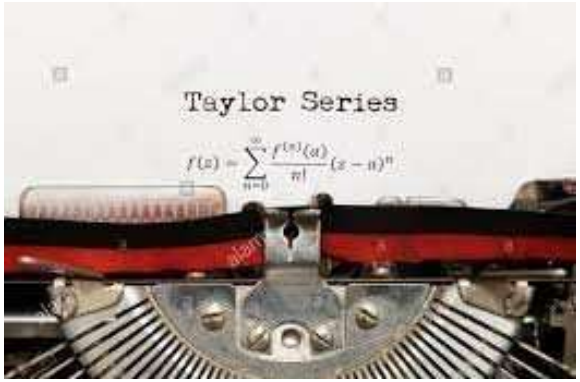
$$\int f(x)dx = C + \sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1}$$

Example 11.2.5

Using the results of Example 11.2.4, find a power series representation for each function. Give the interval of convergence of the resulting series.

$$f(x) = \frac{x^3}{(1 - x^4)^2}$$

$$g(x) = \ln(x + 10)$$



Section 11.3

Taylor Series

Friday, April 22, 2022

Recall: Taylor Polynomials

Let f be a function with $f, f', f'', \ldots,$ and $f^{(n)}$ defined at a. The nth-order Taylor polynomial for f centered at a is

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

p_n matches f in value, slope, and all derivatives up to the nth derivative at a.

Taylor Series

Suppose the function f has derivatives of all orders on an interval centered at the point a. Then, the Taylor series for f centered at a is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

A Taylor series centered at zero is called a Maclaurin series.

Example 11.3.1

Find the Maclaurin series for each of the following functions. Determine the radius and interval of convergence for each.

$$f(x) = e^x$$

$$g(x) = \sin x$$

$$h(x) = \arctan x$$

Example 11.3.2

Find the Taylor series for $f(x) = x - x^3$ centered at $a = -2$ and then determine its radius and interval of convergence.

Example 11.3.3

Find the Taylor series for $f(x) = \frac{1}{\sqrt{x}}$ centered at $a = 9$ and then determine its radius of convergence.

Example 11.3.3

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{-1/2}$	$\frac{1}{3}$
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{2}\left(\frac{1}{3^3}\right)$
2	$-\frac{1}{2}\left(-\frac{3}{2}\right)x^{-5/2}$	$-\frac{1}{2}\left(-\frac{3}{2}\right)\left(\frac{1}{3^5}\right)$
3	$-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2}$	$-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{1}{3^7}\right)$
$\vdots$	$\vdots$	$\vdots$
n		$(-1)^n\frac{1(3)(5)\cdots(2n-1)}{2^n 3^{2n+1}}$

Important Maclaurin Series

These will be given on the formula sheet for all remaining exams.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x| < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad |x| < \infty$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \qquad -1 < x \leq 1$$

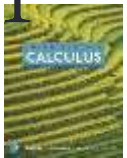
Important Maclaurin Series

These will be given on the formula sheet for all remaining exams.

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad |x| < \infty$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad |x| < \infty$$

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \qquad |x| \leq 1$$



Example 11.3.4

Use known Maclaurin series to find the Maclaurin series for each function and then determine the radius of convergence.

$$f(x) = \sinh x$$

$$g(x) = x \arctan(x^2)$$

Section 11.4

Working with Taylor Series

Monday, April 25, 2022

Recall: Taylor Series

Suppose the function f has derivatives of all orders on an interval centered at the point a. Then, the Taylor series for f centered at a is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

A Taylor series centered at zero is called a Maclaurin series.

Recall: Important Maclaurin Series

These will be given on the formula sheet for all remaining exams.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x| < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad |x| < \infty$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \qquad -1 < x \le 1$$

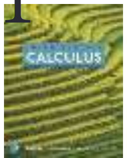
Recall: Important Maclaurin Series

These will be given on the formula sheet for all remaining exams.

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad |x| < \infty$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad |x| < \infty$$

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \qquad |x| \le 1$$



Example 11.4.1

Evaluate the limit using Taylor series.

$$\lim_{x \to 0} \frac{\tan^{-1} x - x}{x^3}$$

Limits using Taylor Series

To evaluate limits as $x \to a$ using Taylor series, it is advisable to use Taylor series centered at a.

To evaluate limits as $x \to \infty$ using Taylor series, use the change of variables $x = \frac{1}{t}$ and evaluate the limit as $t \to 0$ of the resulting expression.

The only limits using Taylor series that will be tested on the final are limits as $x \to 0$.

Example 11.4.2

Differentiate the Maclaurin series for $f(x) = \sin x$ to verify that $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$.

Example 11.4.3

Differentiate the Maclaurin series for the function

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

Identify the function represented by the differentiated series and give its interval of convergence.

Example 11.4.4

Use a Taylor series to approximate the definite integral. Retain as many terms as needed to ensure the error is less than 10^{-4}.

$$\int_0^{0.2} \sin x^2 \, dx$$

Example 11.4.5

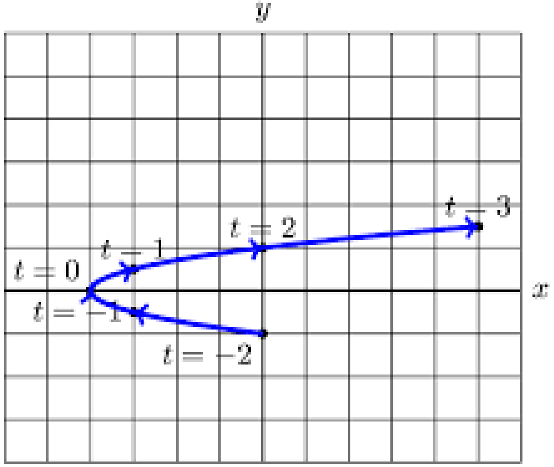
Use an appropriate Taylor series to find the first four nonzero terms of an infinite series that is equal to the number $\sqrt{e}$.

Example 11.4.6

Identify the function represented by each power series.

$$\sum_{k=0}^{\infty} \frac{x^k}{2^k}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{4^k}$$



Section 12.1

Parametric Equations

Wednesday, April 27, 2022

Parametric Equations

When tracing the motion of a particle in the plane, it is often useful to define both its horizontal and vertical position as functions of time.

In other words, we frequently wish to define

$$x = f(t) \text{ and } y = g(t)$$

These are called parametric equations because they both depend on a parameter t.

Example 12.1.1

Sketch the curve defined by the parametric equations

$$x = t^2 - 4 \qquad y = \frac{t}{2}$$

Eliminate the parameter to identify the curve.

Example 12.1.2

Sketch the curve defined by the parametric equations

$$x = \cos t \qquad y = \sin t$$

on the interval $0 \leq t \leq 2\pi$.

Eliminate the parameter to identify the curve.

Example 12.1.3

Identify the curve defined by the parametric equations

$$x = x_0 + at \qquad y = y_0 + bt$$

Parametric Equations of a Line

The equations

$$x = x_0 + at \qquad y = y_0 + bt$$

for $-\infty < t < \infty$ where $x_0, y_0, a,$ and b are constants (with a and b not both zero) define a line passing through the point (x_0, y_0).

If $a \neq 0$, the line has slope $\frac{b}{a}$.

If $a = 0$, the line is vertical.

Example 12.1.4

Find parametric equations for the line segment from the first point to the second point.

$$(1,2) \text{ to } (3,-1)$$

$$(3,-1) \text{ to } (1,2)$$

Example 12.1.5

Find parametric equations for the curve.

The left half of the parabola $y = x^2 + 1$ originating at $(0,1)$.

Tangents of Parametric Curves

Assume $x = f(t)$ and $y = g(t)$ are both differentiable on $a \le t \le b$. Then, the slope of the tangent line at a point corresponding to t is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{g'(t)}{f'(t)}$$

provided $f'(t) \neq 0$.

This is simply a restatement of the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

Example 12.1.6

Consider the parametric curve defined by

$$x = 3t^2 \qquad y = t^3 - t$$

a) Find all points (if any) on the curve where the tangent is horizontal.

b) Find all points (if any) on the curve where the tangent is vertical.

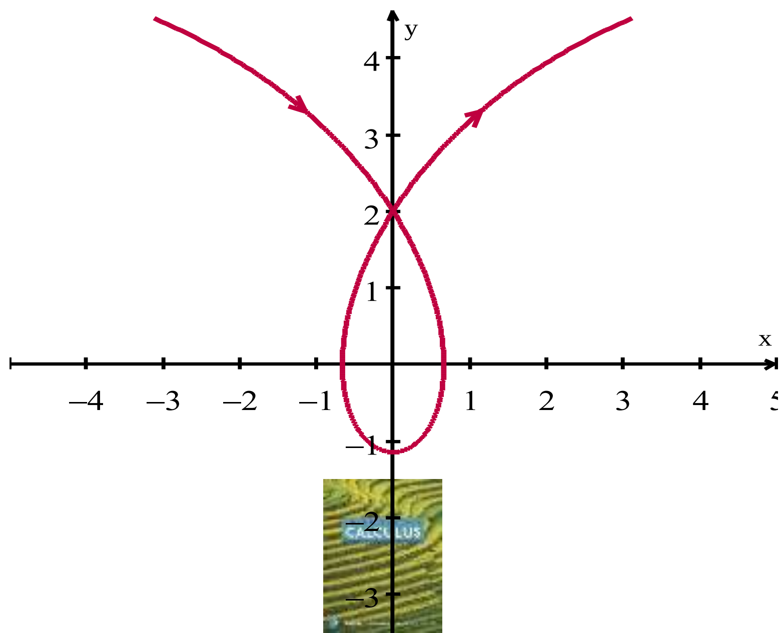
c) Find all points (if any) on the curve where the tangent has a slope of 1/3.

Example 12.1.7

Consider the prolate cycloid given by

$$x = 2t - \pi \sin t \qquad y = 2 - \pi \cos t$$

Find the equations of both tangent lines at $(0,2)$.



Arc Length of a Parametric Curve

Let C be the curve described by

$$x = f(t) \qquad y = g(t) \qquad a \le t \le b$$

where f' and g' are continuous on $a \le t \le b$ and C is traversed exactly once as t increases from a to b. Then, the length of C is

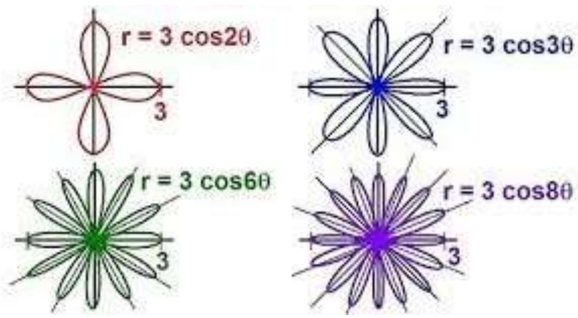
$$L = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\, \mathrm{d}t$$

Example 12.1.8

Find the arc length of each curve on the given interval.

$$x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le \pi$$

$$x = t^{3/2}, \qquad y = 1 + (1 - t)^{3/2}, \qquad 0 \le t \le 1$$



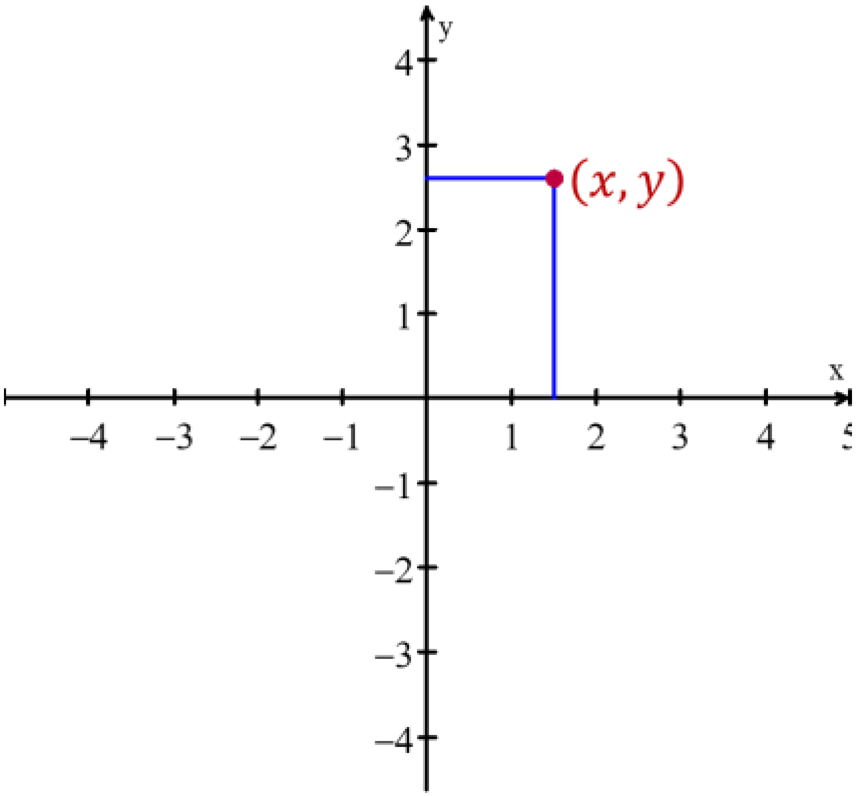
Section 12.2

Polar Coordinates

Friday, April 29, 2022

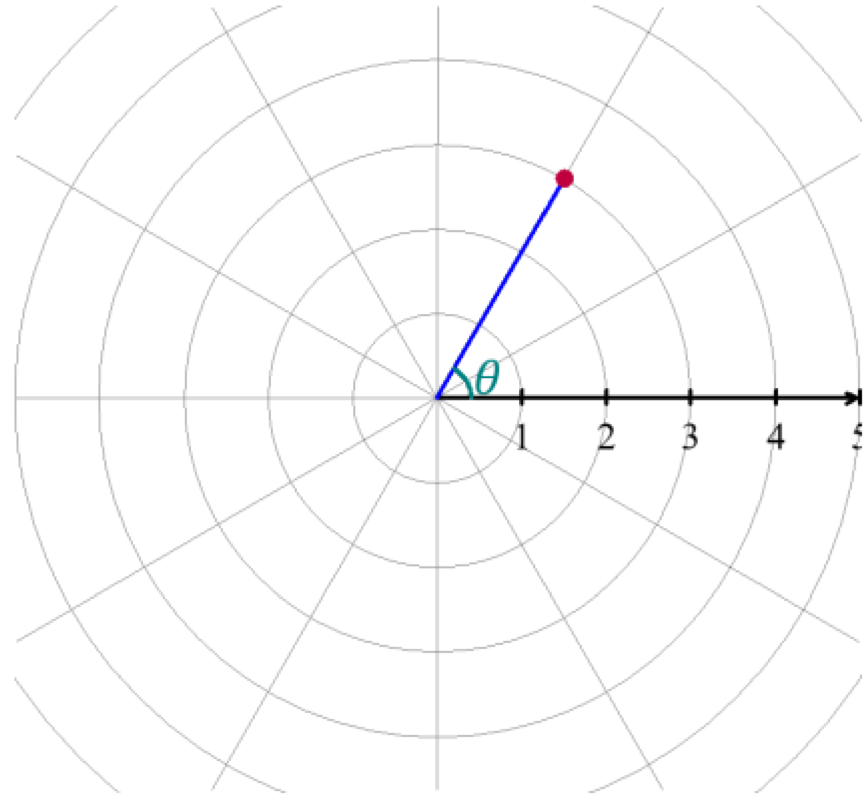
Rectangular and Polar Coordinates

In rectangular coordinates, a point P is represented by an ordered pair (x, y), where x and y are directed distances from the point P to the coordinate axes.



Rectangular and Polar Coordinates

In polar coordinates, a point P is represented by an ordered pair (r,θ) where r is the directed distance from P to the pole (origin) and θ is the directed angle formed by the polar axis and the ray OP.



Note that r and θ may be negative or positive.

Example 12.2.1

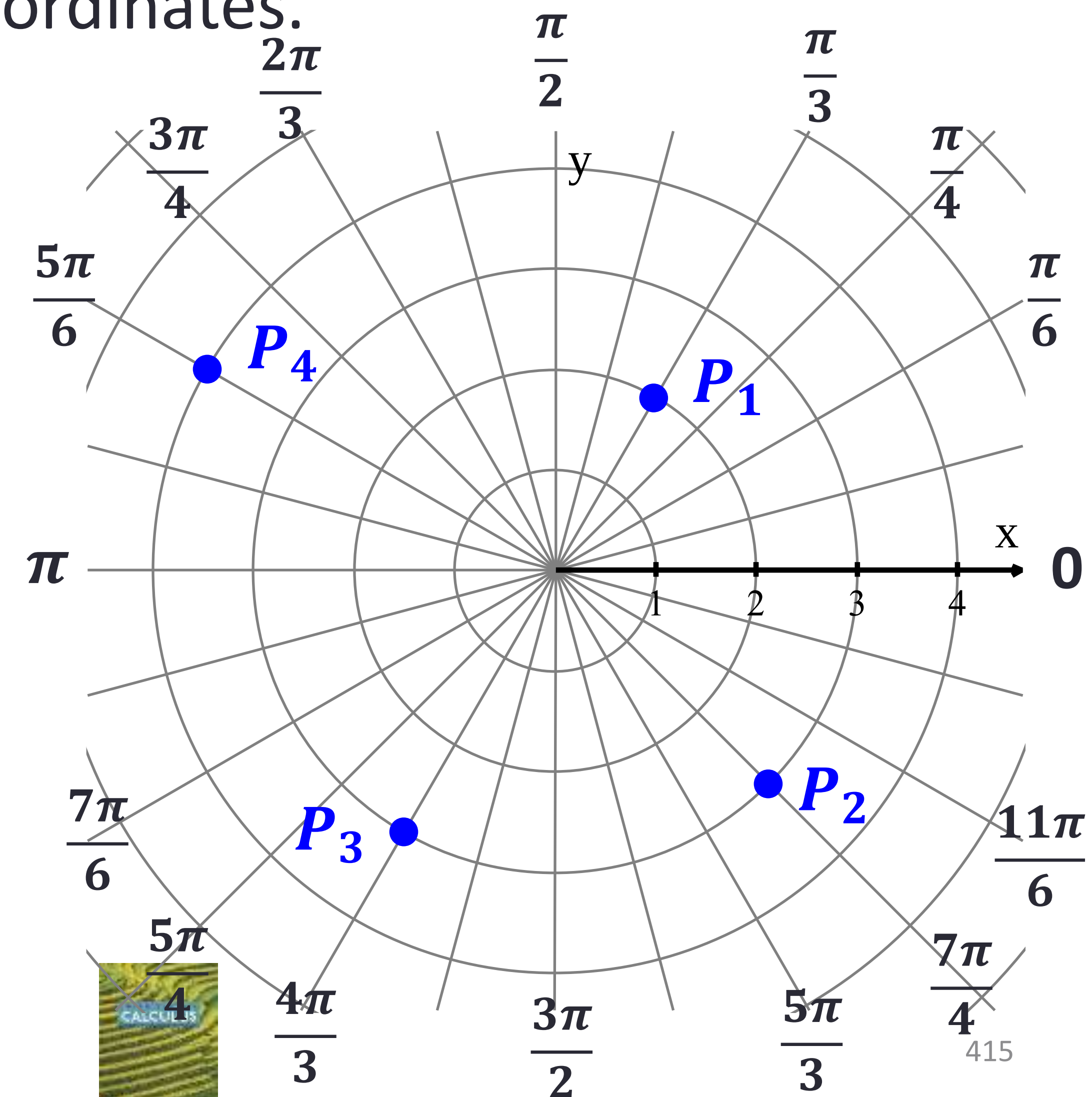
Plot points in polar coordinates.

a) $P_1\left(2, \frac{\pi}{3}\right)$

b) $P_2\left(3, -\frac{\pi}{4}\right)$

c) $P_3\left(-3, \frac{\pi}{3}\right)$

d) $P_4\left(-4, -\frac{\pi}{6}\right)$



Example 12.2.2

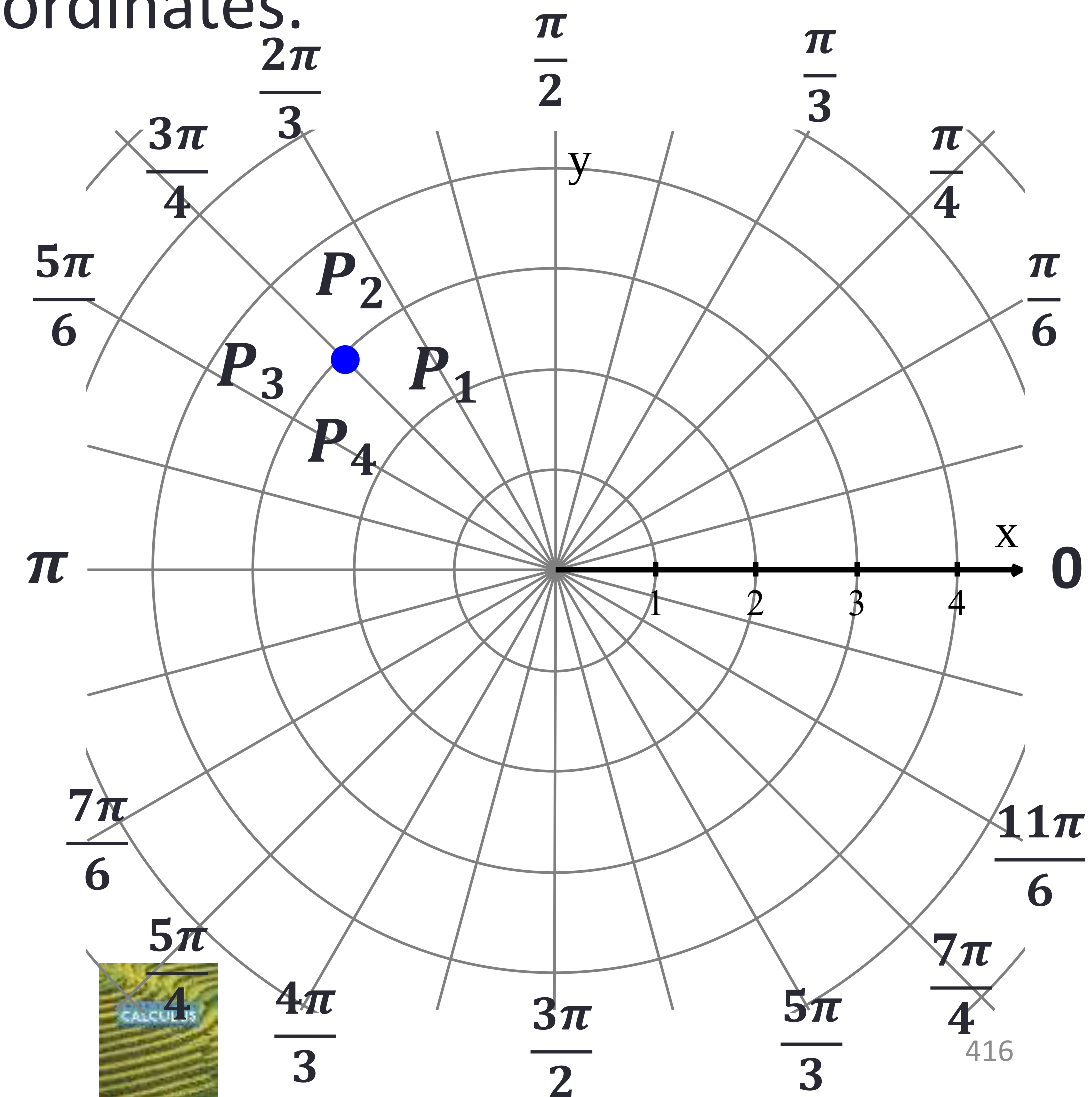
Plot points in polar coordinates.

a) $P_1\left(3, \frac{3\pi}{4}\right)$

b) $P_2\left(-3, -\frac{\pi}{4}\right)$

c) $P_3\left(3, -\frac{5\pi}{4}\right)$

d) $P_4\left(-3, \frac{7\pi}{4}\right)$



Multiple Representations in Polar Coordinates

In polar coordinates, each point may have many representations.

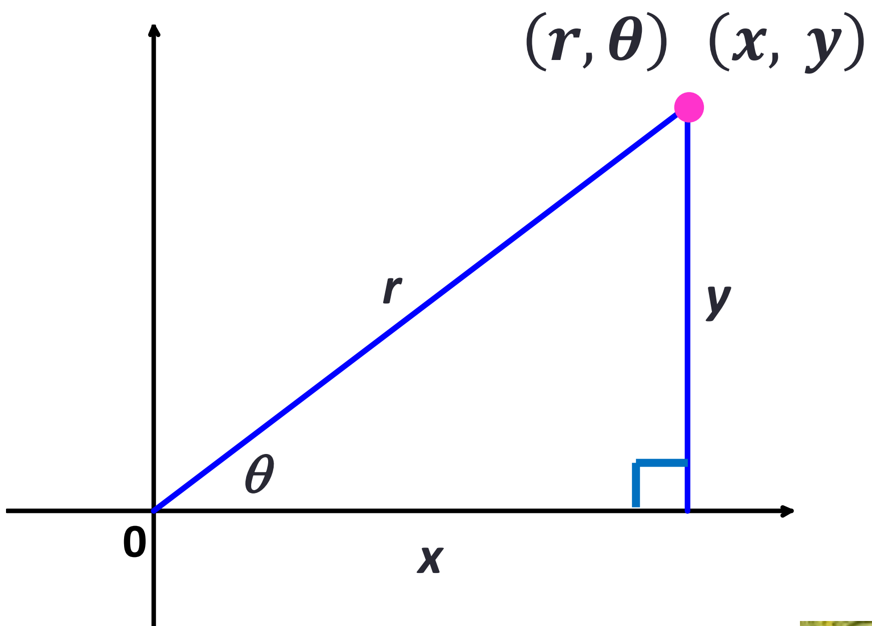
$$(r, \theta) = (r, \theta \pm 2k\pi)$$

$$(r, \theta) = (-r, \theta \pm (2k+1)\pi)$$

The pole is $(0, \theta)$ for any angle θ.

Coordinate Conversion

To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin.



$$x^2 + y^2 = r^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\tan\theta = \frac{y}{x}, x \neq 0$$

Example 12.2.3

Convert from polar to rectangular.

a) $P_1(2, \pi)$

b) $P_2\left(\sqrt{3}, -\frac{\pi}{3}\right)$

c) $P_3\left(-4, \frac{7\pi}{6}\right)$

Example 12.2.4

Convert from rectangular to polar.

a) $P_1(-1, 1)$

b) $P_2(0, 2)$

c) $P_3\left(-1, -\sqrt{3}\right)$

Example 12.2.5

Transform each polar equation into rectangular form.

a) $r = 3$

b) $\theta = \dfrac{\pi}{3}$

c) $2 = r\sin\theta$

d) $r = 4\sin\theta$

Example 12.2.6

Transform each rectangular equation into polar form.

a) $x = 3$

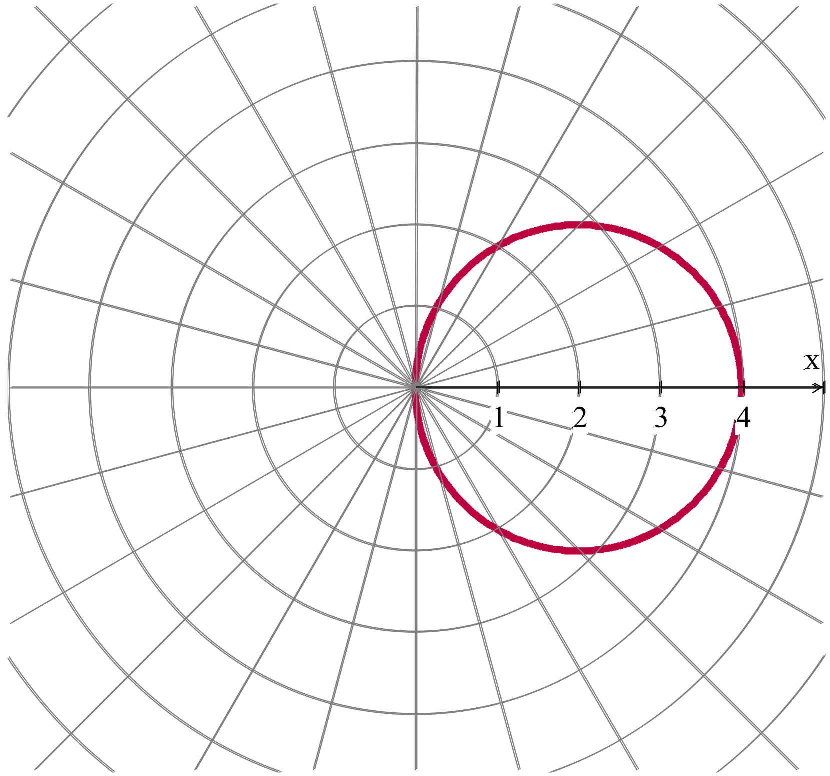
b) $y = -4$

c) $x^2 + y^2 = 5$

d) $y^2 = 2x$

Example 12.2.7

Graph the polar equation $r = 4\cos\theta$.



Circles in Polar Coordinates

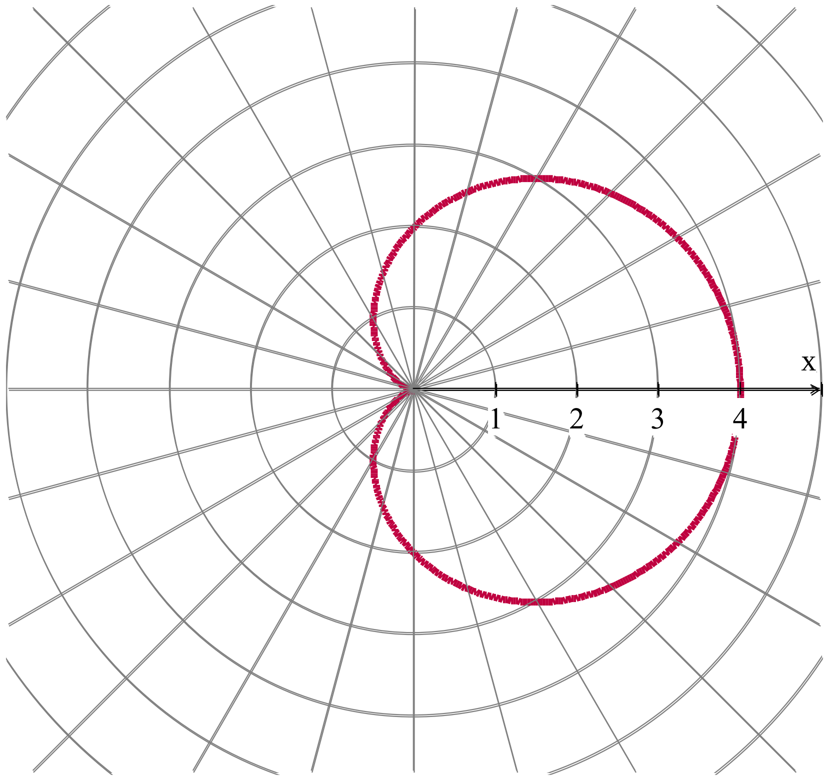
The equation $r = a$ describes a circle of radius $|a|$ centered at $(0,0)$.

The equation $r = 2a\cos\theta + 2b\sin\theta$ describes a circle of radius $\sqrt{a^2+b^2}$ centered at (a,b).

- Note that such a circle passes through the pole.

Example 12.2.8

Graph the polar equation $r = 2 + 2\cos\theta$.

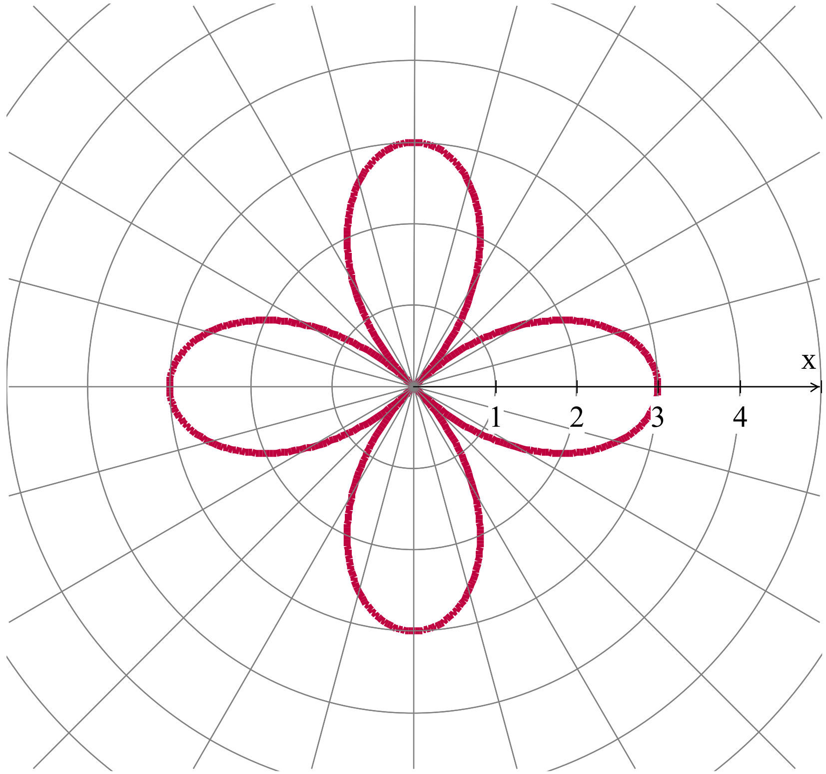


This is an example of 

a cardioid.

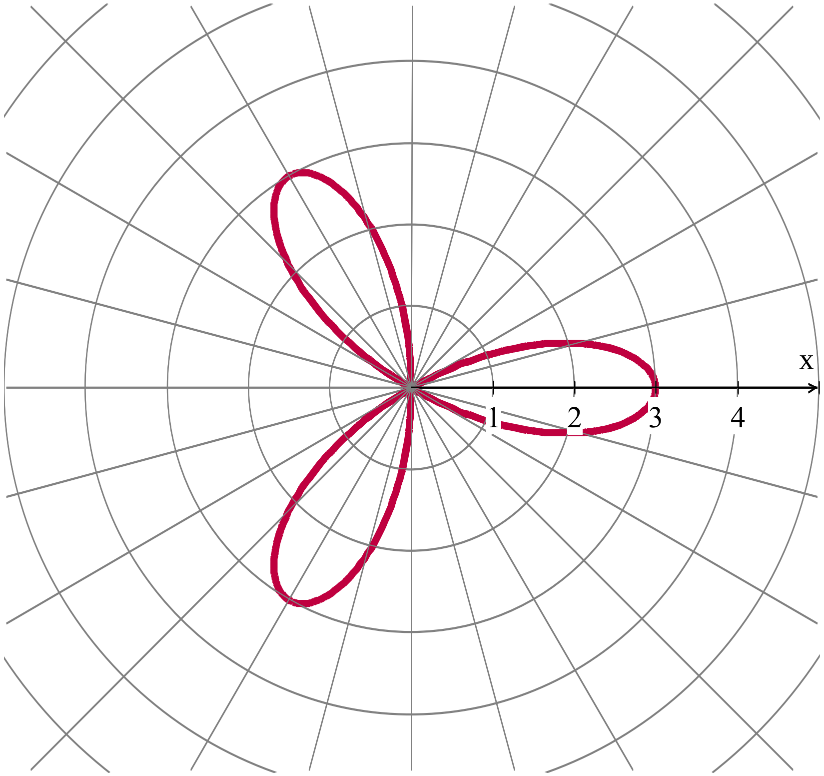
Example 12.2.9

Graph the polar equation $r = 3\cos 2\theta$.



Example 12.2.10

Graph the polar equation $r = 3\cos 3\theta$.



Rose Curves

Rose curves have equations of the form

$$r = a\cos(n\theta)$$

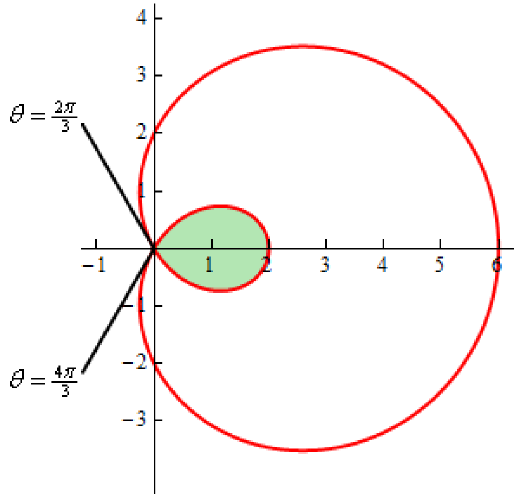
and

$$r = a\sin(n\theta)$$

where a is nonzero and n is an integer.

If n is odd, the rose has n leaves.

If n is even, the rose has $2n$ leaves.



Section 12.3

Calculus in Polar Coordinates

Monday, May 2, 2022

Area of a Circular Sector

The area of a circular sector with radius r and central angle θ (measured in radians) is

$$A = \frac{1}{2}r^2\theta$$

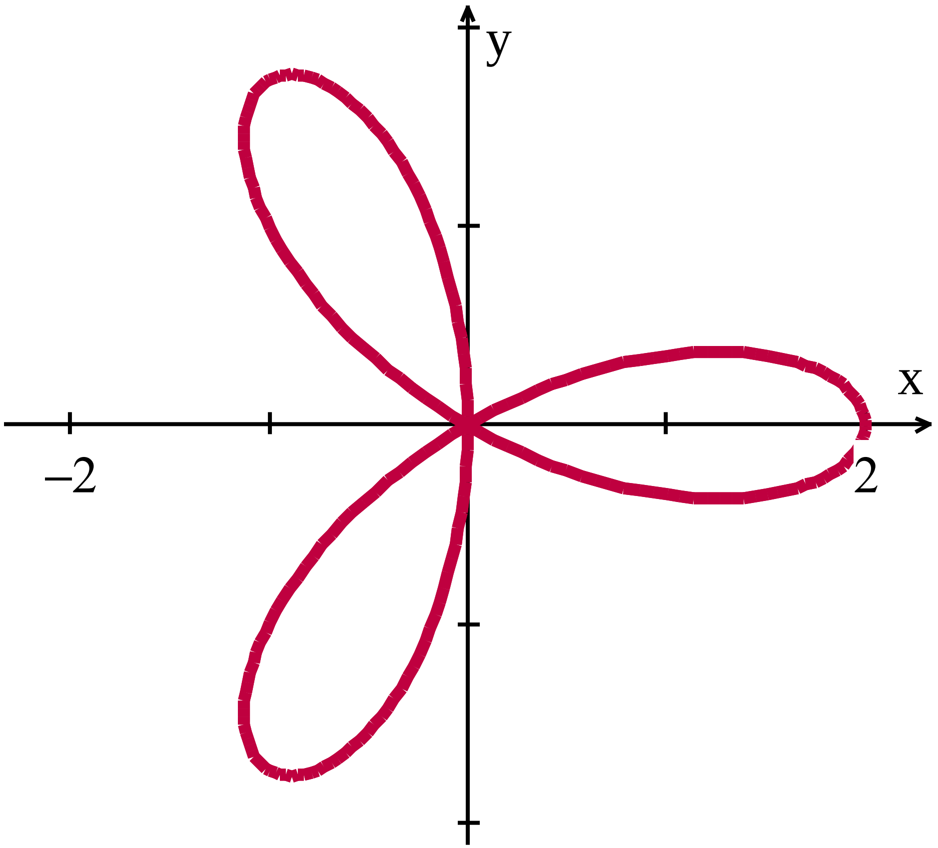
Areas of Simple Polar Regions

Consider the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$. If $f(\theta) \geq 0$ on the interval $\alpha \leq \theta \leq \beta$, then the area of the region is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$

Example 12.3.1

Find the area of one petal of the rose curve $r = 2\cos 3\theta$.



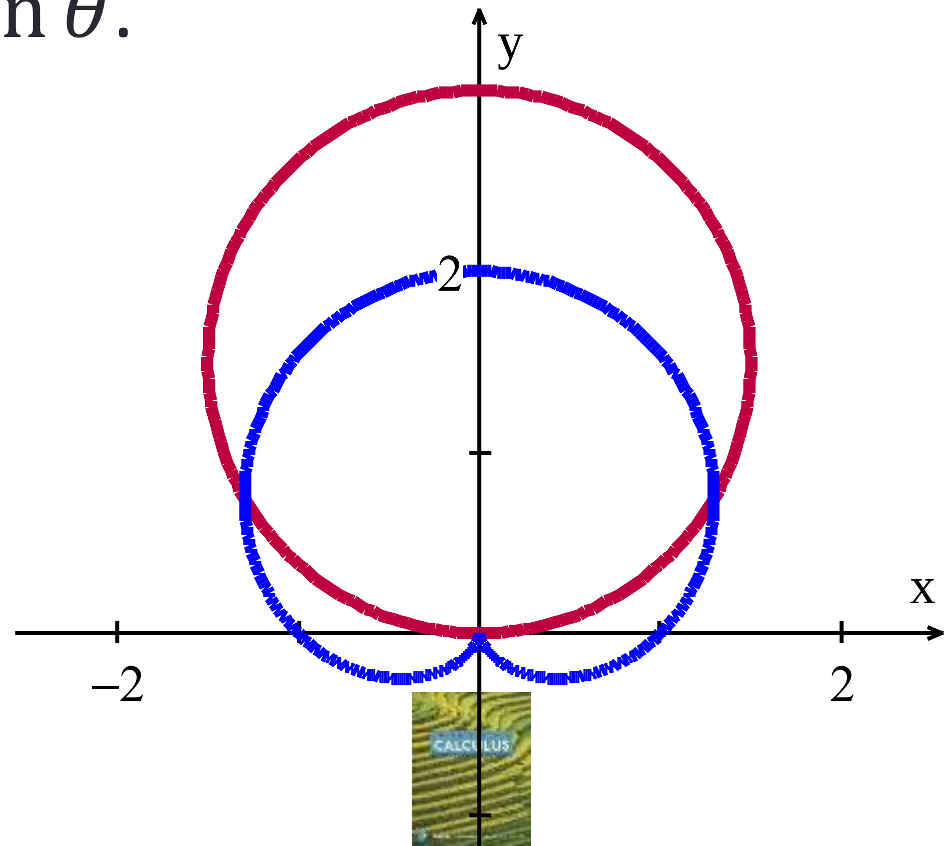
Areas of Polar Regions

Consider the region bounded by the graphs of $r = f(\theta)$ and $r = g(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$. If $f(\theta) \geq g(\theta) \geq 0$ on the interval $\alpha \leq \theta \leq \beta$, then the area of the region is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\left([f(\theta)]^2 - [g(\theta)]^2\right)d\theta$$

Example 12.3.2

Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



Arc Length of a Polar Curve

If the polar curve $r = f(\theta)$ has a continuous derivative on the interval $\alpha \leq \theta \leq \beta$, then the arc length of f on $\alpha \leq \theta \leq \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

Be sure to only traverse the curve once!

Example 12.3.3

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ of the cardioid $r = 2 - 2\cos\theta$.