23. Determine whether the following $(V, +, \cdot)$ are real vector spaces and justify your claims.

(a) $V = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} : x \in \mathbb{R} \right\}$, $\begin{bmatrix} 1 \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ x+y \end{bmatrix}$, $c \cdot \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ cx \end{bmatrix}$;

(b) $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2 + x_2 \\ y_1 + 2 + y_2 \end{bmatrix}$, $c \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$;

(c) $V = (0, \infty)$, $x + y = xy$, $c \cdot x = x^c$;

(d) $V$ the set of all invertible $2 \times 2$-matrices, with usual matrix addition and multiplication of a matrix by a scalar;

(e) $V$ the set of all non-invertible $2 \times 2$-matrices, with usual matrix addition and multiplication of a matrix by a scalar.

24. Which of the following sets are subspaces of $\mathbb{R}^3$? Again, justify your claims.

(a) The set of vectors in $\mathbb{R}^3$ with first component 0;

(b) The set of vectors in $\mathbb{R}^3$ with last component 4;

(c) The set of vectors in $\mathbb{R}^3$ whose components multiplied together gives zero;

(d) The set of vectors in $\mathbb{R}^3$ whose first two components are the same;

(e) The set of all linear combinations of the two vectors $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$;

(f) The set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ whose components satisfy $4a - b + 2c = 0$;

(g) The set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ whose components satisfy $4a - b + 2c - 4 = 0$.

25. Let $V$ be the vector space consisting of all $3 \times 3$-matrices (with usual matrix addition and multiplication of a matrix by a scalar). Find the smallest subspace which contains all symmetric matrices and all lower triangular matrices. What is the largest subspace which is contained in both of these subspaces?

26. Let $V$ be a vector space and let $U_1$ and $U_2$ be subspaces. Prove that $U_1 \cap U_2$ is also a subspace of $V$. How about $U_1 \cup U_2$?