38. Let U and V be vector spaces and $L : U \to V$ be a linear transformation. Show that
    (a) $\mathcal{N}(L) = \{u \in U : \ L(u) = 0\}$ is a subspace of U;
    (b) $\mathcal{R}(L) = \{L(u) : \ u \in U\}$ is a subspace of V.
39. Let P_n be the set of all polynomials of degree smaller or equal to $n \in \mathbb{N}_0$. We know that P_n is a vector space. For $p(t) = \sum_{k=0}^{n} a_k t^k$, define $v(p) = \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^{n+1}$. Let $q(t) = 1 - 4t^2 + 3t^3$.
    (a) For $q \in P_3$, find $v(q)$.
    (b) Find a basis and the dimension of P_3. Find $v(b)$ for each element b in the basis.
    (c) Is $L(p) = p^2$ with $L : P_3 \to P_6$ a linear transformation?
    (d) Is $L(p) = p + q$ with $L : P_3 \to P_3$ a linear transformation?
40. For the following transformations $L : P_3 \to P_n$ do the following: Pick n. Find $L(q)$, where q is given in the previous problem. Find $v(L(q))$. Show that L is a linear transformation. Find a matrix A such that $v(L(p)) = Av(p)$ for all $p \in P_3$. Find $\mathcal{N}(L)$ and $\mathcal{R}(L)$.
    (a) $L(p) = p'$;
    (b) $L(p) = pq$;
    (c) $L(p) = p''$;
    (d) $L(p)$ is the solution of the problem $x' = p$, $x(0) = 0$.
41. A police boat cruising for drug dealers in the Mediterranean uses a linear transformation to encrypt its position $\binom{x_N}{x_E}$ (northern latitude x_N and eastern longitude x_E) and radios the encrypted position to the headquarters in Marseille. The headquarters use another linear transformation and radio their encryption to Paris. Spies find out that, when the boat was at $\binom{42}{6}$, $\binom{54}{156}$ arrived in Marseille and $\binom{576}{258}$ arrived in Paris. Also, they know that, when the boat was at $\binom{39}{15}$, $\binom{516}{183}$ arrived in Paris. As a final piece of information they know that Marseille received a message $\binom{54}{96}$ that was sent to Paris as $\binom{396}{138}$. Now, spies catch the message $\binom{446}{113}$ in Paris. Where is the boat? Which message arrived in Marseille?
42. Work on all of the Review Exercises of Chapter 2 on pages 128–131.