45. Find a basis and the dimension for each of the four fundamental subspaces of $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

46. Give an example (if possible) of a matrix A with the required properties:

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathcal{R}(A)$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix} \in \mathcal{N}(A)$;

(b) $\mathcal{R}(A) = I\!R^4$ and $\mathcal{R}(A^T) = I\!R^3$;

(c) $\mathcal{L}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \mathcal{R}(A)$ and $\mathcal{L}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \mathcal{R}(A^T)$;

(d) $\mathcal{R}(A) = \mathcal{L}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$ and $\mathcal{N}(A) = \mathcal{L}\left(\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}\right)$;

(e) $\mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in I\!R^3 : \ x_1 + 2x_2 + 4x_3 = 0 \right\}$;

(f) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathcal{R}(A^T) \cap \mathcal{N}(A)$.

47. Let $L : I\!R^2 \to I\!R^2$ be a linear transformation of the form $L(x) = Ax$. Determine the image of the triangle with vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ under L, and find $L\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$ and A for the following cases:

(a) $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$;

(b) $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $L\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$;

(c) L reflects the points $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ at the x_2-axis;

(d) L reflects the points $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ at the x_1-axis;

(e) L interchanges $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$;

(f) L reflects the points $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ at the origin;

(g) L multiplies the distances of the points $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the origin by 5 but the directions of the vectors are unchanged (also draw the image of the unit circle under L);

(h) L doubles the distance from the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the origin and triples the distance from the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to the origin (also draw the image of the unit circle under L);

(i) L rotates the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by 45 degrees (this means, L moves this point clockwise around the unit circle until it hits a point of the form $\begin{bmatrix} a \\ a \end{bmatrix}$) and it also rotates the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by 45 degrees;

(j) L rotates $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ by 90 degrees;

(k) L rotates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by 60 degrees;

(l) L replaces the second components of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$ by 0 but leaves the first components unchanged;

(m) L replaces the first components of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$ by 0 and doubles the second components;

(n) L first does (j), then (m);

(o) L first does (f), then (i);

(p) L first does (m), then (l);

(q) L first does (f), then (f);

(r) L first does (j), then (g).

48. A police boat cruising for drug dealers in the Mediterranean uses a linear transformation to encrypt its position $\begin{bmatrix} x_N \\ x_E \end{bmatrix}$ (northern latitude x_N and eastern longitude x_E) and radios the encrypted position to the headquarters in Marseille. The headquarters use another linear transformation and radio their encryption to Paris. Spies find out that, when the boat was at $\begin{bmatrix} 42 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 54 \\ 156 \end{bmatrix}$ arrived in Marseille and $\begin{bmatrix} 576 \\ 258 \end{bmatrix}$ arrived in Paris. Also, they know that, when the boat was at $\begin{bmatrix} 39 \\ 15 \end{bmatrix}$, $\begin{bmatrix} 516 \\ 83 \end{bmatrix}$ arrived in Paris. As a final piece of information they know that Marseille received a message $\begin{bmatrix} 54 \\ 96 \end{bmatrix}$ that was sent to Paris as $\begin{bmatrix} 396 \\ 138 \end{bmatrix}$. Now, spies catch the message $\begin{bmatrix} 446 \\ 113 \end{bmatrix}$ in Paris. Where is the boat? Which message arrived in Marseille?