

14. We define the harmonic numbers as

$$H_n = \sum_{k=1}^n \frac{1}{k}.$$

(a) Find  $H_1, H_2, H_3, H_4, H_5$  as fractions  $\frac{m}{n}$  with  $m, n \in \mathbb{Z}$ .

(b) What is  $H_{n+1} - H_n$  for  $n \in \mathbb{N}$ ?

(c) Prove by mathematical induction that  $H_{2^n} \leq 1 + n$  holds for all  $n \in \mathbb{N}_0$ .

(d) Prove by mathematical induction that  $\sum_{k=1}^n H_k = (n+1)H_n - n$  holds for all  $n \in \mathbb{N}$ .

15. Show that  $\prod_{k=0}^n (1 + x^{(2^k)}) = \frac{1-x^{(2^{n+1})}}{1-x}$  is true for all  $x \neq 1$  and all  $n \in \mathbb{N}$ .

16. Read Section 2.1 in the textbook and work on problems 1–16.

17. Let  $a_n = \frac{1}{n} - \frac{1}{n+1}$ ,  $n \in \mathbb{N}$ .

(a) Find  $a_1, a_2, a_3, a_4$ , and  $a_5$ .

(b) Find  $\sum_{k=1}^5 a_k$ ,  $\prod_{k=1}^5 a_k$ ,  $\sum_{k=1}^{100} a_k$ , and  $\prod_{k=1}^{100} a_k$ .

(c) Let  $m, n \in \mathbb{N}$ . Find  $\sum_{k=m}^n a_k$  and  $\prod_{k=m}^n a_k$ .

(d) Is  $a$  increasing or decreasing?

(e) Is  $a$  bounded above or bounded below?

18. Let  $a_1 = 3$  and  $a_n = 3 + a_{n-1}$  for  $n \in \mathbb{N} \setminus \{1\}$ .

(a) Find  $a_1, a_2, a_3, a_4$ , and  $a_5$ .

(b) Find  $\sum_{k=1}^5 a_k$  and  $\prod_{k=1}^5 a_k$ .

(c) Let  $n \in \mathbb{N}$ . Find  $\sum_{k=1}^n a_k$  and  $\prod_{k=1}^n a_k$ .

(d) Is  $a$  increasing or decreasing?

(e) Is  $a$  bounded above or bounded below?