Exam #2, Math 2051, Dr. M. Bohner | Name: **Solution Key** | Mar 26, 2002.

Part A: Fill in only the boxes and do your work on a separate sheet.

1. Let $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $Z = \{z_1, z_2, z_3\}$, and define $f : X \to Y$, $g : Y \to Z$, $h : Z \to X$ by $f(x_1) = y_1$, $f(x_2) = y_3$, $f(x_3) = y_4$, $f(x_4) = y_2$, $g(y_1) = z_1$, $g(y_2) = z_1$, $g(y_3) = z_3$, $g(y_4) = z_2$, $h(z_1) = x_1$, $h(z_2) = x_2$, and $h(z_3) = x_4$.

   (a) $(h \circ g \circ f)(x_1)$ $\boxed{= \mathbf{x_1}}$

   (b) $(h \circ g \circ f)(x_2)$ $\boxed{= \mathbf{x_4}}$

   (c) $(h \circ g \circ f)(x_3)$ $\boxed{= \mathbf{x_2}}$

   (d) $(h \circ g \circ f)(x_4)$ $\boxed{= \mathbf{x_1}}$

   (e) $g(\{y_1, y_3\})$ $\boxed{= \{\mathbf{z_1}, \mathbf{z_3}\}}$

   (f) $f^{-1}(\{y_1, y_3\})$ $\boxed{= \{\mathbf{x_1}, \mathbf{x_2}\}}$

   (g) $h^{-1}(\{x_3, x_4\})$ $\boxed{= \{\mathbf{z_3}\}}$.

   (h) Is f <u>one-to-one</u>, <u>onto</u>, or <u>invertible</u>? (Underline whatever applies.)

   (i) Is g one-to-one, <u>onto</u>, or invertible? (Underline whatever applies.)

   (j) Is h <u>one-to-one</u>, onto, or invertible? (Underline whatever applies.)

2. Define a relation R on the set $\{1, 2, 3, 4, 5\}$ by $(x, y) \in R$ if $x + y \leq 6$. Underline whatever applies: Is R reflexive, <u>symmetric</u>, antisymmetric, transitive, an equivalence relation, a partial order?

Part B: For the remaining problems, show your work clearly, explaining each step of the proofs. Use only the space allocated for each problem (use separate sheets of paper for additional work).

3. Let $a_n = \frac{1}{n} - \frac{1}{n+1}$, $n \in \mathbb{N}$.

(a) Find $\sum\limits_{k=1}^{100} a_k$ and $\prod\limits_{k=1}^{100} a_k$.

$$\sum_{k=1}^{100} a_k = \sum_{k=1}^{100} \left[\frac{1}{k} - \frac{1}{k+1}\right] = 1 - \frac{1}{101} = \frac{100}{101}$$

$$\prod_{k=1}^{100} a_k = \prod_{k=1}^{100} \frac{1}{k(k+1)} = \frac{1}{(100!)(101!)} = \frac{1}{101(100!)^2}.$$

(b) Is a increasing or decreasing? (Prove your claim!)

Let $n \in \mathbb{N}$. Then

$$\begin{aligned}
\Delta a_n &= a_{n+1} - a_n \\
&= \frac{1}{n+1} - \frac{1}{n+2} - \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{2}{n+1} - \frac{1}{n} - \frac{1}{n+2} \\
&= \frac{2n(n+2) - (n+1)(n+2) - n(n+1)}{n(n+1)(n+2)} \\
&= \frac{2n^2 + 4n - (n^2 + 3n + 2) - (n^2 + n)}{n(n+1)(n+2)} \\
&= \frac{-2}{n(n+1)(n+2)} \\
&< 0,
\end{aligned}$$

So a is decreasing.

(c) Is a bounded above or bounded below? (Prove your claim!)

Let $n \in \mathbb{N}$. Then

$$a_n = \frac{1}{n} - \frac{1}{n+1} \leq \frac{1}{n} \leq 1$$

and

$$a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \geq 0,$$

so a is bounded above by 1 and bounded below by 0.

4. Let $f_1 = 1$, $f_2 = 2$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

(a) Show $f_n < 2^n$ for all $n \in \mathbb{N}$.

First, $f_n < 2^n$ **is clearly true for** $n = 1$ **and** $n = 2$**. Now let** $n \in \mathbb{N}$ **with** $n \geq 2$ **and assume that** $f_k < 2^k$ **is true for all** $1 \leq k \leq n$**. Then**

$$f_{n+1} = f_n + f_{n-1} < 2^n + 2^{n-1} = 2^n(1 + 0.5) < 2^n \cdot 2 = 2^{n+1},$$

so the statement is true for $n + 1$**. By the PMI, it is now true for all** $n \in \mathbb{N}$**.**

(b) Show $f_n > \left(\frac{3}{2}\right)^n$ for all $n \in \mathbb{N} \setminus \{1, 2, 3, 4\}$.

First, $f_n > (3/2)^n$ **is clearly true for** $n = 5$ **and** $n = 6$**. Now let** $n \in \mathbb{N}$ **with** $n \geq 6$ **and assume that** $f_k > (3/2)^k$ **is true for all** $5 \leq k \leq n$**. Then**

$$f_{n+1} = f_n + f_{n-1} > (3/2)^n + (3/2)^{n-1} = (3/2)^n(1+2/3) = (3/2)^n \cdot (5/3) < (3/2)^n \cdot (3/2) = (3/2)^{n+1},$$

so the statement is true for $n + 1$**. By the PMI, it is now true for all** $n \in \mathbb{N}$ **with** $n \geq 5$**.**

5. Define a relation on $\mathbb{Z}$ by $m \sim n$ iff 7 divides $m - n$. Show that $\sim$ is an equivalence relation. Find all equivalence classes.

(a) **Let $m \in \mathbb{Z}$. Then $m - m = 0$ and $7|0$ so that $m \sim m$. Hence $\sim$ is reflexive.**

(b) **Let $m, n \in \mathbb{Z}$ with $m \sim n$. Hence $7|(m-n)$, i.e., $m - n = 7k$ for some $k \in \mathbb{Z}$. Therefore**

$$n - m = -(m - n) = -7k = 7 \cdot (-k),$$

so $7|(n - m)$ and $n \sim m$. Hence $\sim$ is symmetric.

(c) **Let $m, n, p \in \mathbb{Z}$ with $m \sim n$ and $n \sim p$, i.e., $7|(m - n)$ and $7|(n - p)$, i.e., $m - n = 7k$ and $n - p = 7l$ for some $k, l \in \mathbb{Z}$. Therefore**

$$m - p = (m - n) + (n - p) = 7k + 7l = 7(k + l),$$

so $7|(m - p)$, i.e., $m \sim p$. Hence $\sim$ is transitive.

By (a), (b), and (c), $\sim$ is reflexive, symmetric, and transitive, hence an equivalence relation. There are seven equivalence classes

$$[m] = \{7k + m : \ k \in \mathbb{Z}\} \quad \textbf{for} \quad 0 \leq m \leq 6.$$