Section 14.2

Calculus of Vector-Valued Functions
Derivatives and Integrals

Geometrical notions of tangents, normals, arclength, curvature, etc. for space curves will require derivatives and integrals of vector-valued functions. Similarly, kinematic notions like velocity, speed, acceleration, distance traveled, etc. for an object whose positions in space are known will require derivatives and integrals of vector-valued functions.
We say that \( \mathbf{r} \) is \textit{differentiable} at \( t \) provided

\[
\mathbf{r}'(t) = \lim_{{\tau \to 0}} \frac{\mathbf{r}(t + \tau) - \mathbf{r}(t)}{\tau}
\]

exists. If \( \mathbf{r}'(t) \) is not zero, then \( \mathbf{r}'(t) \) is called a \textit{tangent vector} at the point corresponding to \( \mathbf{r}(t) \).
Derivative

\( \mathbf{r} = \langle f, g, h \rangle \) with differentiable \( f, g, h \) satisfies

\[
\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle.
\]

Example: \( \mathbf{r}(t) = \langle te^{-t}, t\ln(t), t\cos(t) \rangle \).

Example: \( \mathbf{r}(t) = \langle \cos(t), \sin(t), ct \rangle \).
The unit tangent vector for a smooth parameterized curve $\mathbf{r}$ is defined to be

$$T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Example: $\mathbf{r}(t) = <t \sin(t), 0, 1 - \cos(t)>$. 
Properties

1. \( \frac{d}{dt} (c) = 0 \)  \hspace{1em} \text{Constant Rule}

2. \( \frac{d}{dt} (u(t) + v(t)) = u'(t) + v'(t) \)  \hspace{1em} \text{Sum Rule}

3. \( \frac{d}{dt} (f(t)u(t)) = f'(t)u(t) + f(t)u'(t) \)  \hspace{1em} \text{Product Rule}

4. \( \frac{d}{dt} (u(f(t))) = u'(f(t))f'(t) \)  \hspace{1em} \text{Chain Rule}

5. \( \frac{d}{dt} (u(t) \cdot v(t)) = u'(t) \cdot v(t) + u(t) \cdot v'(t) \)  \hspace{1em} \text{Dot Product Rule}

6. \( \frac{d}{dt} (u(t) \times v(t)) = u'(t) \times v(t) + u(t) \times v'(t) \)  \hspace{1em} \text{Cross Product Rule}

Example: \( u(t) = \langle 1, t, t^2 \rangle, \; v(t) = \langle t^2, -2t, 1 \rangle, \; r(t) = v(e^t), \; R(t) = u(t) \cdot v(t), \; s(t) = u(t) \times v(t). \)
Indefinite Integral

If \( \mathbf{R}' = \mathbf{r} \), then we call \( \mathbf{R} \) an antiderivative (or indefinite integral) of \( \mathbf{r} \) and write

\[
\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + c
\]

Example: \( \mathbf{r}(t) = \langle e^{3t}, 1/(1+t^2), -(2t)^{-1/2} \rangle \).

Example: \( \mathbf{r}'(t) = \langle t/(t^2+1), te^{-t^2}, -2t/(t^2+4)^{1/2} \rangle \), \( \mathbf{r}(0) = \langle 1, 1.5, -3 \rangle \).
Definite Integral

If \( r = \langle f, g, h \rangle \), where \( f, g, h \) are integrable over \([a, b]\), then the definite integral of \( r \) over \([a, b]\) is defined by

\[
\int_a^b r(t) \, dt = \langle \int_a^b f(t) \, dt, \int_a^b g(t) \, dt, \int_a^b h(t) \, dt \rangle
\]

Example:

\[
\int_1^4 \langle 6t^2, 8t^3, 9t^2 \rangle \, dt
\]