12. Show that the Riemann-Stieltjes integral is unique, if it exists.

13. Prove that $\int f \mathrm{d}g$ is linear in f and g.

14. Find $\int_a^b f \mathrm{d}g$, where g is a constant function.

15. Find $\int_a^b f \mathrm{d}g$, where g is a step function.

16. Define f and g by

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \le x \le 1 \\ 1 & \text{if} \quad 1 < x \le 2 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad 1 \le x \le 2. \end{cases}$$

On which of the intervals $[0,1]$, $[1,2]$, $[0,2]$ is $f \in \mathcal{R}(g)$? Evaluate each of the three integrals, if they exist.

17. Calculate each of the following integrals:

$$\int_0^4 x^2 \mathrm{d}[x], \quad \int_0^\pi x \mathrm{d}\cos x, \quad \int_0^1 x^3 \mathrm{d}x^2, \quad \int_{-1}^2 \sqrt{x+2}\,\mathrm{d}[x].$$

18. For $n \in \mathbb{N}$ and $f \in \mathrm{C}[0,n]$, find $\int_0^n f \mathrm{d}[x]$, and use your result to derive Euler's summation formula:

$$\sum_{k=0}^n f(k) = \int_0^n f(x)\mathrm{d}x + \frac{f(0)+f(n)}{2} + \int_0^n \left(x - [x] - \frac{1}{2}\right) f'(x)\mathrm{d}x.$$

19. Show that refining a partition decreases its upper sum.