39. Show that each compact subset of $\mathbb{R}$ contains a countable dense subset.

40. Suppose $\mathcal{F}$ is a family of functions that are Lipschitz continuous with the same Lipschitz constant. Show that $\mathcal{F}$ is equicontinuous.

41. Suppose $\mathcal{F}$ is a family of differentiable functions on $[a, b]$ such that the derivatives are uniformly bounded. Show that $\mathcal{F}$ is equicontinuous.

42. Suppose $\mathcal{F}$ is a family of uniformly bounded and Riemann integrable functions on $[a, b]$. Show that the family of functions defined by $\int_a^x f(t)\mathrm{d}t$, where $f \in \mathcal{F}$, is equicontinuous.

43. For a function f defined on $[0, 1]$ we introduce the nth Bernstein polynomial as

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Show the following:

(a) $B_n(1; x) \equiv 1$;

(b) $B_n(x; x) = x$;

(c) $B_n(x^2; x) = x^2 + \frac{x(1-x)}{n}$.

44. Show that the polynomials in the "Weierstraß Approximation Theorem" can be chosen as Bernstein polynomials.

45. Find the third Bernstein polynomial for $\sin(\pi x/2)$.

46. Find the fourth Bernstein polynomial for $\sqrt{x}$.

47. Show that the "Weierstraß Approximation Theorem" does not hold if $[a, b]$ is replaced by (a, b).